Combinatorics in sublattices of invariant subspaces

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Sublattices of invariant subspaces

$\mathbb{F}$ a field, $f \in End(\mathbb{F}^n)$, $V \subset \mathbb{F}^n$ subspace.

$Z(f) = \{g \in End(\mathbb{F}^n) : fg = gf\}$, $Z^*(f) = \{g \in Aut(\mathbb{F}^n) : fg = gf\}$.

- V is an **invariant subspace** if $f(V) \subset V$, Inv(f).
- $V \in$ Inv(f) is **characteristic** if $g(V) \subset V$, $\forall g \in Z^*(f)$, Chinv(f).
- V is a **hyperinvariant** if $g(V) \subset V$, $\forall g \in Z(f)$, Hinv(f).

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Proposition

- $\mathrm{Hinv}(f) \subset \mathrm{Chinv}(f) \subset \mathrm{Inv}(f)$.
- $\mathrm{Inv}(f), \mathrm{Hinv}(f), \mathrm{Chinv}(f)$ *lattices.*
- $\mathrm{Hinv}(f), \mathrm{Chinv}(f)$ *finite.*

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- Hinv$(f) \subset$ Chinv$(f) \subset$ Inv(f)*.*
- Inv(f), Hinv(f), Chinv(f) *lattices.*
- Hinv(f), Chinv(f) *finite.* # Hinv(f) ? # Chinv(f) ?

$\mathrm{Hinv}(J)$ (Fillmore-Herrero-Longstaff, 1977)

$f \equiv J \in \mathcal{M}_n(\mathbb{F})$,

$$\mathrm{Hinv}(J) = \{V \subset \mathbb{F}^n : TV \subseteq V, \ \ \forall T \in Z(J)\}$$

- J **nilpotent Jordan**, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m > 0$ Segre characteristic
- $$\begin{array}{ccc} S \subset \mathbb{N}^m & \xrightarrow{\textbf{bijection}} & \mathrm{Hinv}(J) \\ (k_1, \ldots, k_m) & \to & V = V(k_1, \ldots, k_m) \end{array}$$

$$\left.\begin{array}{r} k_1 \geq k_2 \geq \cdots \geq k_m \geq 0 \\ \alpha_1 - k_1 \geq \alpha_2 - k_2 \geq \cdots \geq \alpha_m - k_m \geq 0 \end{array}\right\} \ (k_1, \ldots, k_m) \ \textbf{hypertuple}$$

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- $\#(\text{Hinv}(J)) = (\alpha_m + 1)(\alpha_{m-1} - \alpha_m + 1) \ldots (\alpha_1 - \alpha_2 + 1)$

Chinv(f) (Shoda 1930, Astuti-Wimmer 2009, Mingueza-Montoro-Pacha 2018)

$$\mathrm{Chinv}(J) = \{V \subset \mathbb{F}^n : TV \subseteq V, \forall T \in Z^*(J)\}$$

$$\boxed{\mathrm{Chinv}(J) = \mathrm{Hinv}(J) \cup (\mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J))}$$

$$\#(\mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J))?$$

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- $\mathbb{F} \neq GF(2) \to \mathrm{Hinv}(J) = \mathrm{Chinv}(J)$ (AW 2009)
- $\mathbb{F} = GF(2)$

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- $\mathbb{F} = GF(2)$
  - J nilpotent Jordan (AW 2009)
  - Characterization of $\mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J) \neq \emptyset$ (Shoda 1930)
  - $X \in \mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J) \leftrightarrow$ special $X = Y \oplus Z$ (MMP 2014)

Theorem (Mingueza-Montoro-Pacha, 2014)

$$X \in \mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J)$$

$$\Updownarrow$$

there exists a **chartuple** *b such that*

?

$$X = Y \oplus Z,$$

Y is a **hyperinvariant subspace** *and Z is a* **minext** *associated to b.*

? ?

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Lemma

$$\#(char(t)) = \sum_{\{i_1,\dots,i_t\}\subset\Omega} \alpha_{i_t}(\alpha_{i_{t-1}} - \alpha_{i_t} - 1)\cdots(\alpha_{i_1} - \alpha_{i_2} - 1)$$

$$\left.\begin{array}{r} b_{i_1} > b_{i_2} > \cdots > b_{i_t} > 0 \\ \alpha_{i_1} - b_{i_1} > \alpha_{i_2} - b_{i_2} > \cdots > \alpha_{i_t} - b_{i_t} \geq 0 \end{array}\right\} \quad (b_{i_1}, \dots, b_{i_t}) \quad \textbf{chartuple}$$

Hyperinvariant subspaces Y associated to b

$b = (b_{i_1}, \ldots, b_{i_t})$ chartuple. Y **hyperinvariant associated to** b is

$V(k_1, \ldots, k_{i_1-1}, b_{i_1} - 1, k_{i_1+1}, \ldots, k_{i_2-1}, b_{i_2} - 1, k_{i_2+1}, \ldots, k_{i_t-1}, b_{i_t} - 1, k_{i_t+1}, \ldots, k_m)$,

such that the following tuple is also hyperinvariant

$$V(k_1, \ldots, k_{i_1-1}, b_{i_1}, k_{i_1+1}, \ldots, k_{i_2-1}, b_{i_2}, k_{i_2+1}, \ldots, k_{i_t-1}, b_{i_t}, k_{i_t+1}, \ldots, k_m).$$

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Problem

Given $\alpha = (\alpha_1, \ldots, \alpha_r)$, $c_1 \geq c_r$, $\alpha_1 - c_1 \geq \alpha_r - c_r$, *find the number of hyperinvariant subspaces*

$$V(c_1, k_2, \ldots, k_{r-1}, c_r)$$

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Problem

Given $\alpha = (\alpha_1, \ldots, \alpha_r)$, $c_1 \geq c_r$, $\alpha_1 - c_1 \geq \alpha_r - c_r$, *find the number of hyperinvariant subspaces*

$$V(c_1, k_2, \ldots, k_{r-1}, c_r)$$

Theorem (Mingueza-Montor-R. 2017)

$$\#\{V(c_1, k_2, \ldots, k_{r-1}, c_r)\}$$

is given by the coefficient of degree $c_1 - c_r + 1$ *of the polynomial* $(d_i = \alpha_i - \alpha_{i+1})$

$$\psi_r(x) = (1 + x + \cdots + x^{d_1}) \cdots (1 + x + \cdots + x^{d_{r-1}}).$$

$\alpha = (11, 9, 7, 5, 4, 3, 3)$, jumps $d = (2, 2, 2, 1, 1, 0)$, $d_i = \alpha_i - \alpha_{i+1}$

$(9, k_2, k_3, k_4, k_5, k_6, 2)$???

10							
$c_1 = 9$	5						
8		**1**					
7		**4**					
6			**1**				
5			**3**				
4				**1**			
3				**2**	**1**		
$c_r = 2$					**1**	**1**	1
1							
i	1	2	3	4	5	6	7

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$(9, k_2, k_3, k_4, k_5, k_6, 2)$???

10							
$c_1 = 9$	5						
8	13	**1**					
7	22	**4**					
6	24	8	**1**				
5	22	10	**3**				
4	13	8	4	**1**			
3	5	4	3	**2**	**1**		
$c_r = 2$	1	1	1	1	**1**	**1**	1
1							
i	1	2	3	4	5	6	7

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5	22	10	**3**				
4	13	8	4	**1**			
3	5	4	3	**2**	**1**		
$c_r = 2$	1	1	1	1	**1**	**1**	1
1							
i	1	2	3	4	5	6	7

d_i	$i \setminus j$	2	3	4	5	6	7	8	9
	b_7	1	0	0	0	0	0	0	0
0									
	k_6	1	0	0	0	0	0	0	0
1									
	k_5	1	1	0	0	0	0	0	0
1									
	k_4	1	2	1	0	0	0	0	0
2									
	k_3	1	3	4	3	1	0	0	0
2									
	k_2	1	4	8	10	8	4	1	0
2									
	$b_1 - 1$	1	5	13	22	24	22	13	5

$$\alpha = (18, 15, 10, 8, 5), \qquad \text{jumps } d = (3, 5, 2, 3), \qquad d_i = \alpha_i - \alpha_{i+1}$$

$$(11, k_2, k_3, k_4, 3) \quad ???$$

	1	2	3	4	5
12					
$c_1 = 11$	38	**6**			
10	43	**9**			
9	43	**11**			
8	38	**12**	**1**		
7	30	11	**2**		
6	19	9	**3**	**1**	
5	10	6	**3**	**1**	
4	4	3	**2**	**1**	
$c_r = 3$	1	1	1	1	1
2					
i	1	2	3	4	5

		3	4	5	6	7	8	9	10	11
	k_5	1	0	0	0	0	0	0	0	0
3										
	k_4	1	1	1	1	0	0	0	0	0
2										
	k_3	1	2	3	3	2	1	0	0	0
5										
	k_2	1	3	6	9	11	12	11	9	6
3										
	k_1	1	4	10	19	30	38	43	43	38

$H = \text{allowed jump} - \textit{jump} = 3+5+2+3-(11-3) = 13-8 = 5$ *looseness*.

$$\binom{4}{1}\binom{3}{3}+\binom{4}{2}\binom{2}{1}+\binom{3}{1}\binom{3}{2}+\binom{3}{1}\binom{3}{1}+\binom{1}{1}\binom{3}{1}+\binom{1}{1}\binom{3}{0}=38.$$

Generalization of the Pascal matrix

$$
\left.\begin{array}{ccccccc}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\begin{bmatrix} i-1 \\ 0 \end{bmatrix} & \begin{bmatrix} i-1 \\ 1 \end{bmatrix} & \dots & \begin{bmatrix} i-1 \\ \Delta_{i-1} \end{bmatrix} & \dots & 0 & \dots & 0 \\
\begin{bmatrix} i \\ 0 \end{bmatrix} & \begin{bmatrix} i \\ 1 \end{bmatrix} & \dots & \begin{bmatrix} i \\ \Delta_{i-1} \end{bmatrix} & \dots & \begin{bmatrix} i \\ \Delta_i \end{bmatrix} & \dots & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & 0 \\
\begin{bmatrix} r-1 \\ 0 \end{bmatrix} & \begin{bmatrix} r-1 \\ 1 \end{bmatrix} & \dots & \begin{bmatrix} i \\ \Delta_{i-1} \end{bmatrix} & \dots & \begin{bmatrix} r-1 \\ \Delta_i \end{bmatrix} & \dots & \begin{bmatrix} r-1 \\ \Delta_{r-1} \end{bmatrix}
\end{array}\right|
\begin{array}{c}
\psi_1(x) \\ \vdots \\ \psi_i(x) \\ \psi_{i+1}(x) \\ \vdots \\ \psi_r(x)
\end{array}
$$

$$\Delta_i = d_1 + \dots + d_i, \quad i = 1, \dots, r-1$$

Generalization of the Pascal matrix

$$\left.\begin{array}{ccccccccc}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\begin{bmatrix} i-1 \\ 0 \end{bmatrix} & \begin{bmatrix} i-1 \\ 1 \end{bmatrix} & \ldots & \begin{bmatrix} i-1 \\ \Delta_{i-1} \end{bmatrix} & \ldots & 0 & \ldots & 0 \\
\begin{bmatrix} i \\ 0 \end{bmatrix} & \begin{bmatrix} i \\ 1 \end{bmatrix} & \ldots & \begin{bmatrix} i \\ \Delta_{i-1} \end{bmatrix} & \ldots & \begin{bmatrix} i \\ \Delta_i \end{bmatrix} & \ldots & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & 0 \\
\begin{bmatrix} r-1 \\ 0 \end{bmatrix} & \begin{bmatrix} r-1 \\ 1 \end{bmatrix} & \ldots & \begin{bmatrix} i \\ \Delta_{i-1} \end{bmatrix} & \ldots & \begin{bmatrix} r-1 \\ \Delta_i \end{bmatrix} & \ldots & \begin{bmatrix} r-1 \\ \Delta_{r-1} \end{bmatrix}
\end{array}\right| \begin{array}{c} \psi_1(x) \\ \vdots \\ \psi_i(x) \\ \psi_{i+1}(x) \\ \vdots \\ \psi_r(x) \end{array}$$

$$\Delta_i = d_1 + \ldots + d_i, \quad i = 1, \ldots, r-1$$

Remark

- $$\begin{aligned} &\psi_1(x) = 1, \\ &\psi_{i+1}(x) = \psi_i(x)(1 + x + x^2 + \cdots + x^{d_{r-i+1}}) \end{aligned}$$

Generalization of the Pascal matrix

$$
\left.\begin{array}{cccccccc|c}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \psi_1(x) \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots \\
\begin{bmatrix} i-1 \\ 0 \end{bmatrix} & \begin{bmatrix} i-1 \\ 1 \end{bmatrix} & \dots & \begin{bmatrix} i-1 \\ \Delta_{i-1} \end{bmatrix} & \dots & 0 & \dots & 0 & \psi_i(x) \\
\begin{bmatrix} i \\ 0 \end{bmatrix} & \begin{bmatrix} i \\ 1 \end{bmatrix} & \dots & \begin{bmatrix} i \\ \Delta_{i-1} \end{bmatrix} & \dots & \begin{bmatrix} i \\ \Delta_i \end{bmatrix} & \dots & 0 & \psi_{i+1}(x) \\
\vdots & \vdots & & \vdots & & \vdots & & 0 & \vdots \\
\begin{bmatrix} r-1 \\ 0 \end{bmatrix} & \begin{bmatrix} r-1 \\ 1 \end{bmatrix} & \dots & \begin{bmatrix} i \\ \Delta_{i-1} \end{bmatrix} & \dots & \begin{bmatrix} r-1 \\ \Delta_i \end{bmatrix} & \dots & \begin{bmatrix} r-1 \\ \Delta_{r-1} \end{bmatrix} & \psi_r(x)
\end{array}\right.
$$

$$\Delta_i = d_1 + \ldots + d_i, \quad i = 1, \ldots, r-1$$

Remark

- $$\psi_1(x) = 1,$$
  $$\psi_{i+1}(x) = \psi_i(x)(1 + x + x^2 + \cdots + x^{d_{r-i+1}})$$
- $\psi_4(x) = (1 + x + x^2 + x^3)(1 + x + x^2)(1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3)$

$\equiv \begin{bmatrix} 1 & 4 & 10 & 19 & 29 & 38 & 43 & 43 & 38 & 29 & 19 & 10 & 4 & 1 \end{bmatrix} \rightarrow \psi_4(x)(11 - 3 + 1)$

Minext subspaces Z associated to b

$\alpha = (\alpha_1, \ldots, \alpha_m)$, $\quad b = (b_{i_1}, \ldots, b_{i_t})$ chartuple. Let

$$z_j = J^{\alpha_{i_j} - b_{i_j}} u_{i_j}, \qquad 1 \leq j \leq t$$

Z is a **minext subspace** associated to b if:

1. $z \in Z \Rightarrow z = z_{i_1} + \cdots + z_{i_p}$, $p \leq t$.
2. $z_j \notin Z$, for $j = 1, \ldots, t$.
3. Each z_j appears as a summand of some $z \in Z$:

$$\dim(\mathrm{span}\{z_1, \ldots, \hat{z}_j, \ldots, z_t\} + Z) = t, \qquad \forall j = 1, \ldots, t.$$

Cardinality of d-dimensional minext subspaces

Theorem (Mingueza-Montor-R. 2017)

$b = (b_{i_1}, \dots, b_{i_t})$ chartuple, $N_d(t)$ number of d-dimensional minext subspaces associated to b. Then,

$$N_d(t) = \binom{t}{d}_2 - \sum_{k=1}^{d} (-1)^{k+1} \binom{t}{k} \binom{t-k}{d-k}_2 - \sum_{k=d+1}^{t-1} \binom{t}{k} N_d(k)$$

$N_d(k) = 0$ *if* $k \leq d$, $\quad N_d(d+1) = 1$.

Remark

$\#\{b - \text{minext of dimension } d\}$ depends on d, t but not on α.

Example

Number of d-dimensional minext subspaces for $2 \leq t \leq 10$, $1 \leq d \leq 8$.

$t \backslash d$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0
4	1	9	1	0	0	0	0	0
5	1	35	35	1	0	0	0	0
6	1	115	445	115	1	0	0	0
7	1	357	3985	3985	357	1	0	0
8	1	1085	31157	87705	31157	1085	1	0
9	1	3271	229579	1583607	1583607	229579	3271	1
10	1	9831	1646185	26048985	62907909	26048985	1646185	9831

Thanks for your attention!