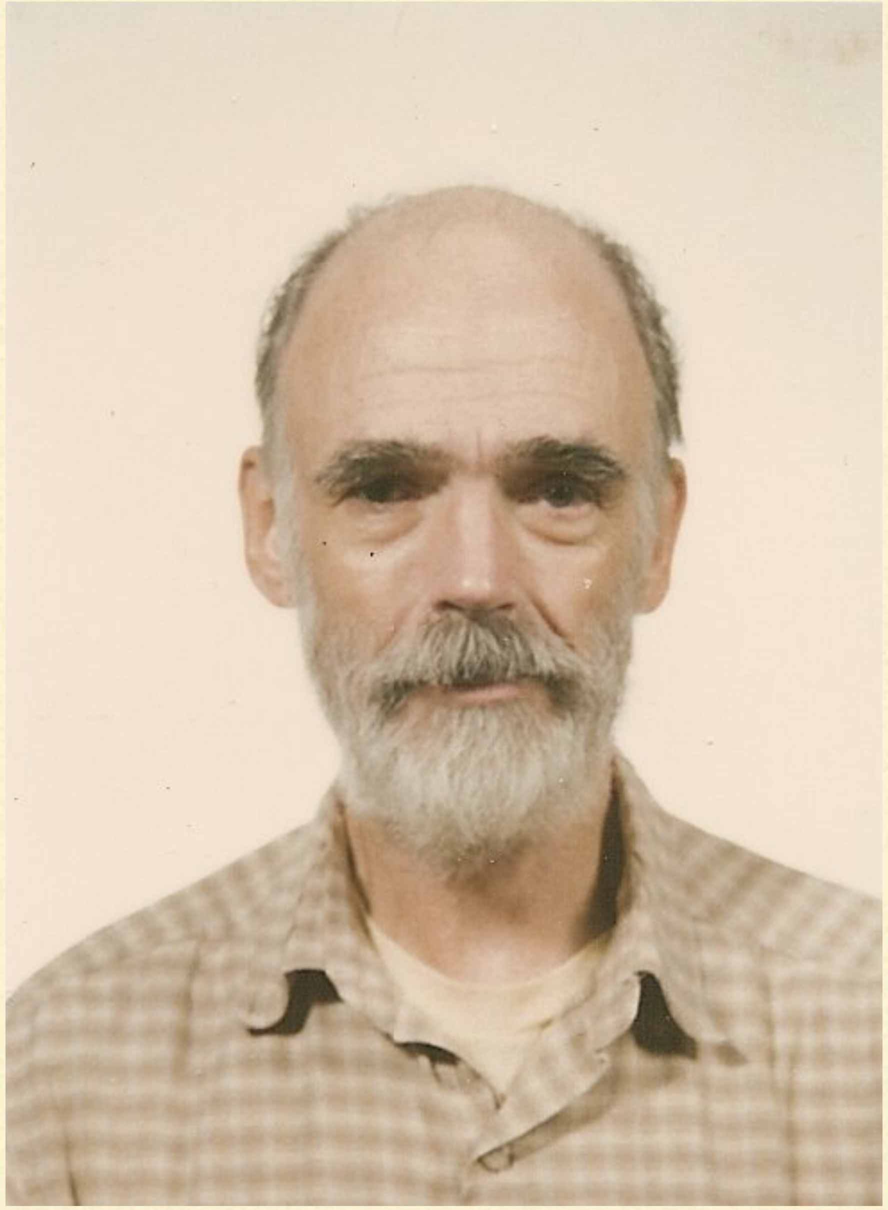
Variations on some themes in the work of Donald G. Higman



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The Mathematics of Donald Gordon Higman

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1. Introduction

Donald Gordon Higman (born 20 September 1928 in Vancouver, B.C., Canada)—an architect of important theories in finite groups, representation theory, algebraic combinatorics, and geometry and a longtime faculty member at the University of Michigan (1960–1998)—died after a long illness on 13 February 2006.

Don left a significant legacy of mathematical work and personal impact on many mathematicians. A committee was formed in 2006 to work with the *Michigan Mathematical Journal* and create a memorial. The contributors have some mathematical closeness to Don. Several of Don's fifteen doctoral students are included in this group. The breadth of topics and quality of the writing is impressive. For example, the article of Broué is especially direct in examining the impact of one of Higman's basic results in representation theory (the "Higman criterion").

Don Higman was a serious intellectual who had the manner of a kind uncle or concerned friend. He worked broadly in algebra and combinatorics. He thought deeply about the ideas in his mathematical sphere, and his style was to seek the essence of a theory. His work had great influence on future developments. This is exemplified by one of his theorems in permutation groups, as related by Peter Neumann: Don's "fundamental observation that a permutation group is primitive if and only if all its nontrivial orbital graphs are connected changed the character of permutation group theory. It's a simple thing, but it introduces a point of view that allowed lovely simplifications and extensions of the proofs of many classical theorems due to Jordan, Manning, and Wielandt."

Len Scott relates Don's reaction to a John Thompson lecture, around 1968, at a conference at the University of Illinois. This was not long after the discovery of the *Higman–Sims sporadic simple group*. Thompson expressed agreement with Jacques Tits's "heliocentric view of the universe, with the general linear group as the sun, and these sporadic groups as just asteroids." Len happened to be on the same elevator with Don, shortly after the lecture, when one of the participants asked Don what he thought of the heliocentric model. Don's reply was, "Well, it hurts your eyes to look at the sun all the time."

The elevator passengers had a good laugh, and it really was a marvelous line. But, reflecting further, not only can we see a part of Don's personality and humor here, but also some of his identity as a mathematician and even some of his place in mathematical history.

"Don Higman was a serious intellectual who had the manner of a kind uncle or concerned friend"

"his style was to seek the essence of a theory"

"He was repeatedly attracted to the idea of elegant, simple explanations and to finding axiom systems"

“DGH asked me to send him copies of my preprints. I cannot explain how overwhelmed I felt by his asking me to do this. I really felt like I was unworthy, that I should presume to send this great man something that I had written.” [C. Praeger]

At Thompson lecture: “[J. Tits’s] heliocentric view of the universe, with the general linear group as the sun, and these sporadic groups as just asteroids.” When asked what he thought of this model, DGH replied: “Well, it hurts your eyes to look at the sun all the time.”

Highlights from ’88 to ’95+

- Study of CCs of small type

$\begin{bmatrix} 2 & 2 \\ & 2 \end{bmatrix}$ = symmetric designs

$\begin{bmatrix} 2 & 2 \\ & 3 \end{bmatrix}$ = quasi-symmetric designs

$\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$ = strongly regular designs

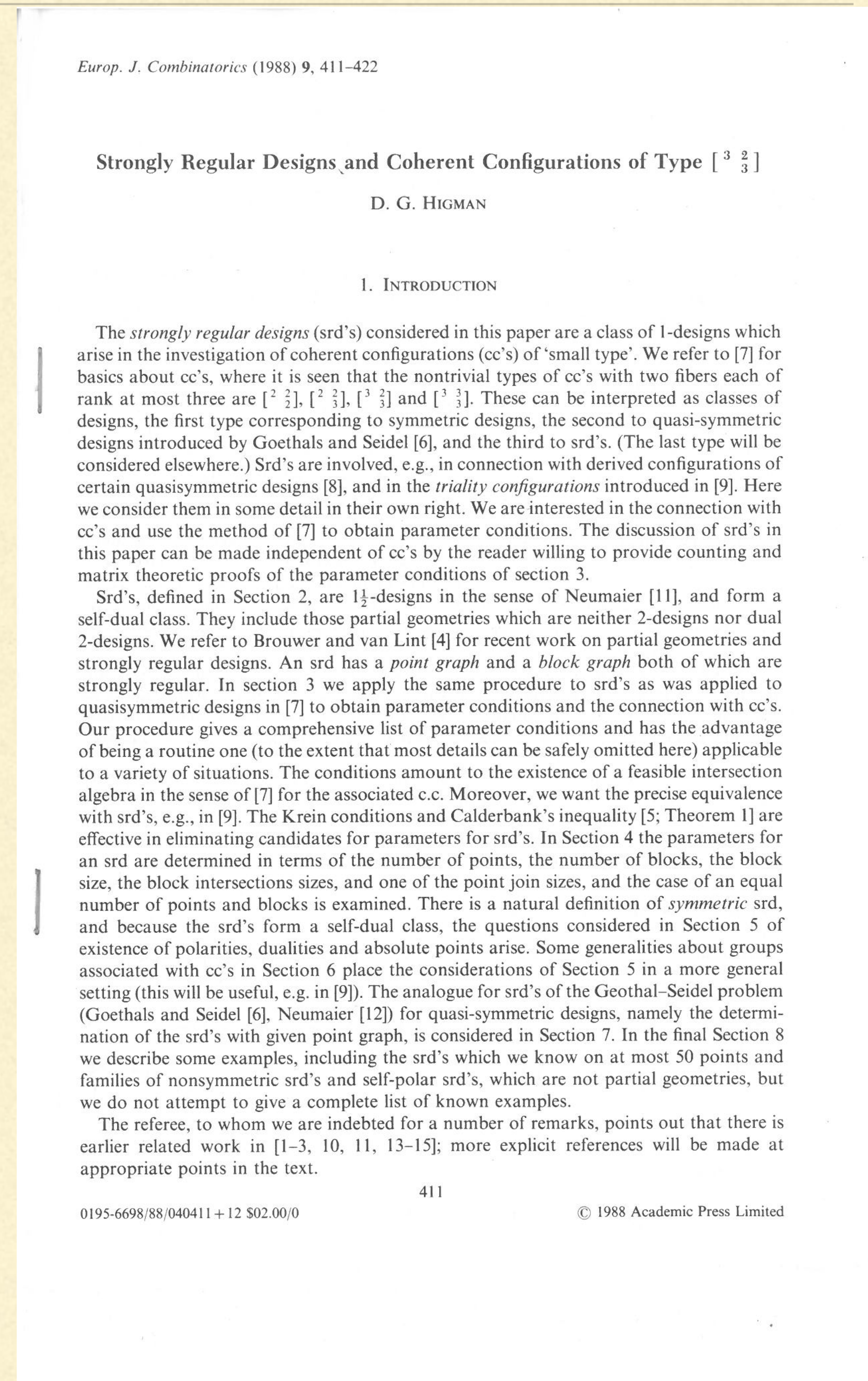
$\begin{bmatrix} 3 & 3 \\ & 3 \end{bmatrix}$ = strongly regular designs of the second kind

- Generalizations: uniformity, weights and t-graphs, semi-coherent configurations, relation configurations

Strongly regular designs

point graph, block graph SRGs;
point-block incidence

- known in design theory literature (quasi-symmetric special PBIBD)
- analysis of parameters
- CC is the WL-closure of the flags
- rank 5 schemes if symmetric, rank 10 CCs otherwise
- further work: Hobart, Klin & Reichard, Hanaki



Europ. J. Combinatorics (1988) **9**, 411–422

Strongly Regular Designs and Coherent Configurations of Type $\left[\begin{smallmatrix}3 & 2\\ & 3\end{smallmatrix}\right]$

D. G. Higman

1. Introduction

The *strongly regular designs* (srd's) considered in this paper are a class of 1-designs which arise in the investigation of coherent configurations (cc's) of 'small type'. We refer to [7] for basics about cc's, where it is seen that the nontrivial types of cc's with two fibers each of rank at most three are $\left[\begin{smallmatrix}2 & 2\\ & 2\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2 & 2\\ & 3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}3 & 2\\ & 3\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}3 & 3\\ & 3\end{smallmatrix}\right]$. These can be interpreted as classes of designs, the first type corresponding to symmetric designs, the second to quasi-symmetric designs introduced by Goethals and Seidel [6], and the third to srd's. (The last type will be considered elsewhere.) Srd's are involved, e.g., in connection with derived configurations of certain quasisymmetric designs [8], and in the *triality configurations* introduced in [9]. Here we consider them in some detail in their own right. We are interested in the connection with cc's and use the method of [7] to obtain parameter conditions. The discussion of srd's in this paper can be made independent of cc's by the reader willing to provide counting and matrix theoretic proofs of the parameter conditions of section 3.

Srd's, defined in Section 2, are $1\frac{1}{2}$-designs in the sense of Neumaier [11], and form a self-dual class. They include those partial geometries which are neither 2-designs nor dual 2-designs. We refer to Brouwer and van Lint [4] for recent work on partial geometries and strongly regular designs. An srd has a *point graph* and a *block graph* both of which are strongly regular. In section 3 we apply the same procedure to srd's as was applied to quasisymmetric designs in [7] to obtain parameter conditions and the connection with cc's. Our procedure gives a comprehensive list of parameter conditions and has the advantage of being a routine one (to the extent that most details can be safely omitted here) applicable to a variety of situations. The conditions amount to the existence of a feasible intersection algebra in the sense of [7] for the associated c.c. Moreover, we want the precise equivalence with srd's, e.g., in [9]. The Krein conditions and Calderbank's inequality [5; Theorem 1] are effective in eliminating candidates for parameters for srd's. In Section 4 the parameters for an srd are determined in terms of the number of points, the number of blocks, the block size, the block intersections sizes, and one of the point join sizes, and the case of an equal number of points and blocks is examined. There is a natural definition of *symmetric* srd, and because the srd's form a self-dual class, the questions considered in Section 5 of existence of polarities, dualities and absolute points arise. Some generalities about groups associated with cc's in Section 6 place the considerations of Section 5 in a more general setting (this will be useful, e.g. in [9]). The analogue for srd's of the Geothal–Seidel problem (Goethals and Seidel [6], Neumaier [12]) for quasi-symmetric designs, namely the determination of the srd's with given point graph, is considered in Section 7. In the final Section 8 we describe some examples, including the srd's which we know on at most 50 points and families of nonsymmetric srd's and self-polar srd's, which are not partial geometries, but we do not attempt to give a complete list of known examples.

The referee, to whom we are indebted for a number of remarks, points out that there is earlier related work in [1–3, 10, 11, 13–15]; more explicit references will be made at appropriate points in the text.

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Strongly regular decomposition

SRG Γ_0, vertices partitioned into SRGs Γ_1, Γ_2

- main technique interlacing of eigenvalues
- qs-design when Γ_0 primitive, Γ_1 complete or null
- SRD except when Γ_0 conference type
- strong parameter conditions
- constructions (3-designs, MOLS)
- hemisystem: SR decomposition of point graph of GQ(q^2, q)

Strongly Regular Graphs with Strongly Regular Decomposition

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

1. INTRODUCTION AND PRELIMINARY RESULTS

The title refers to strongly regular graphs Γ_0 which admit a partition $\{X_1, X_2\}$ of the vertex set such that each of the induced subgraphs Γ_1 and Γ_2 on X_1 and X_2 respectively is strongly regular, a clique, or a coclique. A central role is played by the design D having point set X_1, block set X_2, and incidence given by adjacency in Γ_0. If Γ_1 is a clique or a coclique and Γ_0 is primitive, D must be a quasisymmetric design. If Γ_1 and Γ_2 are both strongly regular, D is a strongly regular design in the sense of D. G. Higman [14], except possibly when Γ_0 is the graph of a regular conference matrix. Conversely, a quasisymmetric or strongly regular design with suitable parameters gives rise to a strongly regular graph with strongly regular decomposition. Moreover, if Γ_0 and Γ_1 are strongly regular with suitable parameters, then Γ_2 must be strongly regular too. We give several examples and some nonexistence results. We include a table of all feasible parameter sets up to 300 vertices. For most of the cases in the table existence or nonexistence is settled. Some of the results in this paper are old, due to M. S. Shrikhande [17], W. G. Bridges and M. S. Shrikhande [3], and W. H. Haemers [13].

We mainly use eigenvalue techniques. We need results on interlacing eigenvalues (see [13]). Two sequences $\rho_1 \geqslant \cdots \geqslant \rho_n$ and $\sigma_1 \geqslant \cdots \geqslant \sigma_m$

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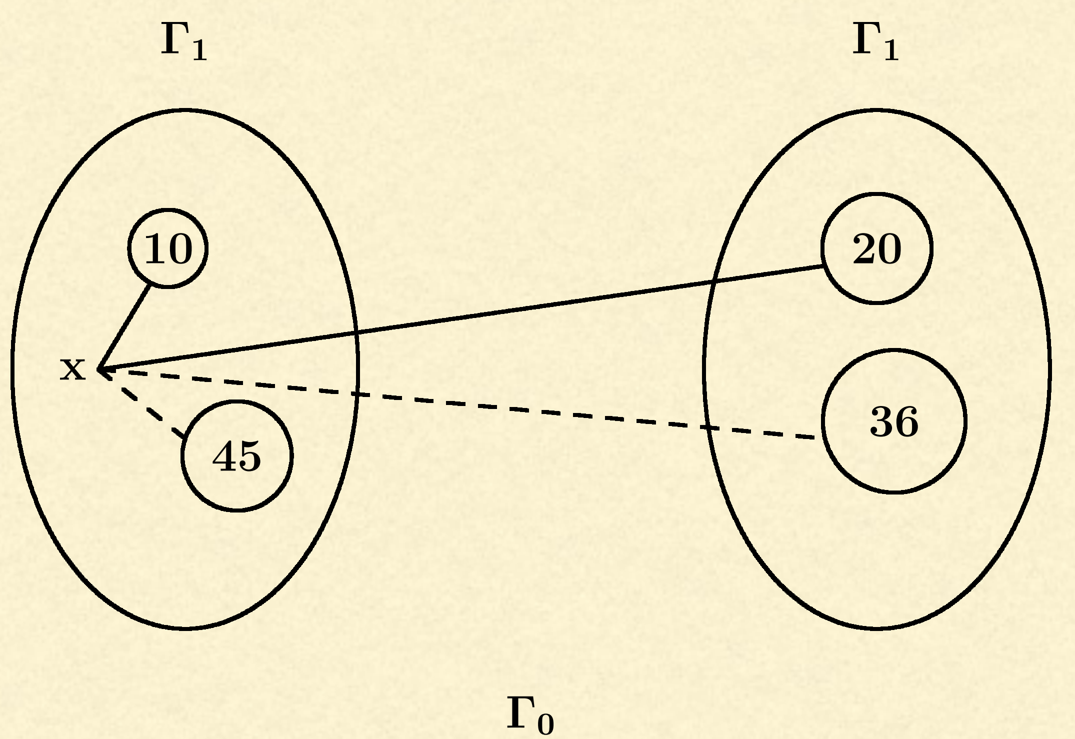
A strongly regular decomposition

- srg(112,30,2,10) split into two srg(56,10,0,2)



- rank 5 CC on 112 (symmetric SRD)
- explicitly treated in Higman & Haemers, van Dam et. al.

Imprimitive, symmetric, rank 5 schemes

- Classified by rank/corank of a parabolic
- 3/2 (class I)
  - fusion to rank 4 wreath product
  - *uniform* property extends SRD to *system of uniformly linked SRDs*
  - examples from classical groups

NORTH-HOLLAND

Rank 5 Association Schemes and Triality

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Dedicated to J. J. Seidel

Submitted by Willem Haemers

ABSTRACT

Symmetric association schemes having a parabolic of rank 3 and corank 2 satisfying a certain uniformity condition can be interpreted on the one hand as linked strongly regular designs (analogous to linked symmetric designs) and on the other as geometries. Classical triality associated with the groups $O_8^+(q)$ provides a family of examples with rank 3, corank 2 parabolics and sporadic examples are associated with the groups $L_3(4)$, $U_6(2)$ and $U_3(5)$. The triality examples and the $U_3(5)$ example are flag-transitive viewed as geometries. Results include a characterization on the level of parameters of the triality schemes leading to a characterization of the schemes using results of Cameron and Drake, and a proof (as a Cayley exercise) of simple connectivity of the example associated with $U_3(5)$.

0. INTRODUCTION

Motivated by examples associated with classical triality and related group-theoretical phenomena, we investigate imprimitive, symmetric rank 5 association schemes. As indicated in Section 1, the problem reduces to the consideration of three classes I, II, III according to the possibilities for the ranks of certain residues and quotients. Sections 2 through 8 are concerned with class I, which contains the examples equivalent to symmetric strongly regular designs and the examples related to triality. The intersection matrices and character-multiplicity tables are described in Section 2. In Section 4 we observe that *uniform* class I schemes as defined in Section 3 are equivalent to

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Uniform schemes

- Extends defn of uniform to higher rank
- Fully developed by van Dam et al: independently studied in context of dismantlable cometric schemes

  “It turns out that many of these examples are cometric Q-antipodal”

- Uniform equivalent to dismantlable

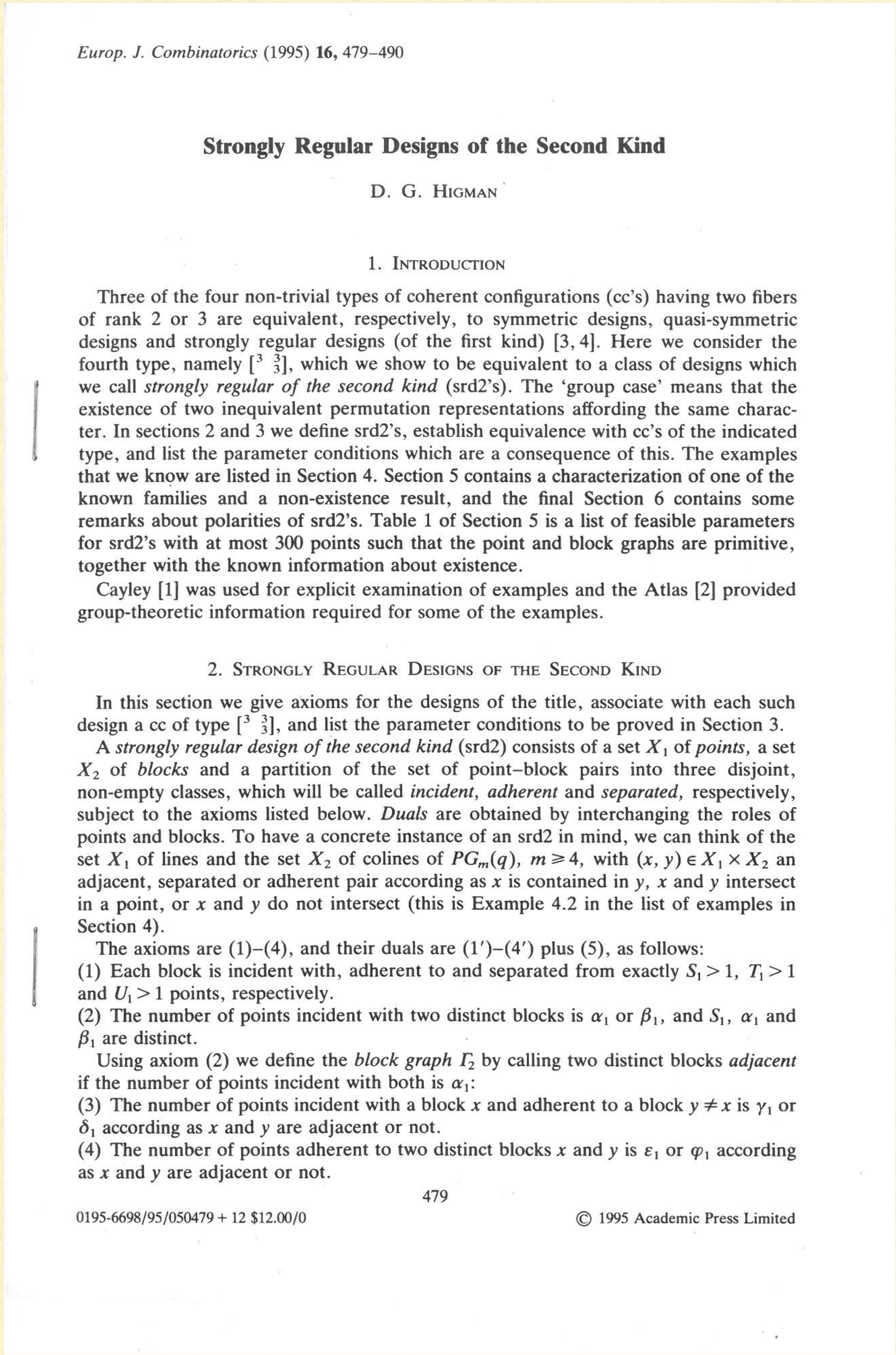
Uniform association schemes.

D. G. Higman

The class of uniform (t,p,m)-schemes considered in the present paper is an extension of the class of uniform schemes of rank 5 encountered in [DGH5]. The schemes in this class are symmetric association schemes characterised by the existence of a parabolic E satisfying certain regularity conditions (conditions (A), (B) and (C) of section 3 below). The number of residues modulo E is t+1 and the rank of the scheme is r+m, where m is the rank of the schemes induced on the residue classes. The conditions imposed on E are motivated by the examples listed in section 7. Associated with each scheme in the class is a coherent configuration having the residues modulo E as its fibers. Determination of the irreducible representations of this configuration leads to a description of the character table of the scheme, and working out the relations between the characters and the intersection numbers produces an effective list of feasibility conditions. Each basic relation of the scheme which is not contained in E can be interpreted as the incidence relation of a geometry having the residue classes modulo E as the types of varieties. In section 7 two flag-transitive geometries are added to the list of flag-transitive geometries obtained in this way in [DGH5]. Each of the examples of a uniform (t,p,m)-scheme listed in section 7 is afforded by a transitive action of a group of the form $\Gamma = G \cdot S_{t+1}$ with G simple and S_{t+1} acting faithfully on G. The group G acts as a group of automorphism of each of the geometries, flag-transitively on one of the geometries in the flag-transitive cases. The group S_{t+1} permutes the types. Since G is simple, $t \le 3$; in fact t = 2 except for the (3,3,6)-scheme mentioned above. For each example at least one amalgam is given on which S_{t+1} has an action as a group of automorphims equivalent to its action on the lattice of subsets of a (t+1)-set.

SRDs of second kind

- Motivation from geometry; examples from groups
- 3 point-block relations (think line and plane)
- Analysis of parameters



Europ. J. Combinatorics (1995) **16,** 479–490

Strongly Regular Designs of the Second Kind

D. G. Higman

1. Introduction

Three of the four non-trivial types of coherent configurations (cc's) having two fibers of rank 2 or 3 are equivalent, respectively, to symmetric designs, quasi-symmetric designs and strongly regular designs (of the first kind) [3, 4]. Here we consider the fourth type, namely $[\begin{smallmatrix}3 & 3\\ & 3\end{smallmatrix}]$, which we show to be equivalent to a class of designs which we call *strongly regular of the second kind* (srd2's). The 'group case' means that the existence of two inequivalent permutation representations affording the same character. In sections 2 and 3 we define srd2's, establish equivalence with cc's of the indicated type, and list the parameter conditions which are a consequence of this. The examples that we know are listed in Section 4. Section 5 contains a characterization of one of the known families and a non-existence result, and the final Section 6 contains some remarks about polarities of srd2's. Table 1 of Section 5 is a list of feasible parameters for srd2's with at most 300 points such that the point and block graphs are primitive, together with the known information about existence.

Cayley [1] was used for explicit examination of examples and the Atlas [2] provided group-theoretic information required for some of the examples.

2. Strongly Regular Designs of the Second Kind

In this section we give axioms for the designs of the title, associate with each such design a cc of type $[\begin{smallmatrix}3 & 3\\ & 3\end{smallmatrix}]$, and list the parameter conditions to be proved in Section 3.

A *strongly regular design of the second kind* (srd2) consists of a set X_1 of *points,* a set X_2 of *blocks* and a partition of the set of point–block pairs into three disjoint, non-empty classes, which will be called *incident, adherent* and *separated,* respectively, subject to the axioms listed below. *Duals* are obtained by interchanging the roles of points and blocks. To have a concrete instance of an srd2 in mind, we can think of the set X_1 of lines and the set X_2 of colines of $PG_m(q)$, $m \geq 4$, with $(x, y) \in X_1 \times X_2$ an adjacent, separated or adherent pair according as x is contained in y, x and y intersect in a point, or x and y do not intersect (this is Example 4.2 in the list of examples in Section 4).

The axioms are (1)–(4), and their duals are (1′)–(4′) plus (5), as follows:

(1) Each block is incident with, adherent to and separated from exactly $S_1 > 1$, $T_1 > 1$ and $U_1 > 1$ points, respectively.

(2) The number of points incident with two distinct blocks is α_1 or β_1, and S_1, α_1 and β_1 are distinct.

Using axiom (2) we define the *block graph* Γ_2 by calling two distinct blocks *adjacent* if the number of points incident with both is α_1:

(3) The number of points incident with a block x and adherent to a block $y \neq x$ is γ_1 or δ_1 according as x and y are adjacent or not.

(4) The number of points adherent to two distinct blocks x and y is ε_1 or φ_1 according as x and y are adjacent or not.

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Regular weights

Motivation:

- extending regular two-graph to regular t-graph
- combinatorial setting for monomial representation
- $(1_H)^G$ gives perm repn with centralizer alg a CA
- $(\text{linear}_H)^G$ gives monomial repn; centralizer alg a weighted CA

WEIGHTS AND t-GRAPHS

D.G.HIGMAN

To Professor J.Tits on the occasion of his 60th birthday

0. INTRODUCTION

Let X be a finite set and let U_t be the group of complex t-th roots of unity. In [4,5,6 and 7] the term weight is used to mean a function from $X \times X$ to $\{0\} \cup U_t$ which, when viewed as a matrix, is hermitian with unit diagonal. Regularity of a weight on an association scheme, or more generally, on a coherent configuration (cc), is defined in terms of a suitable coboundary operator. The original motivation for these definitions (which are repeated below) was to provide a generalization to monomial representations of the combinatorial description of the centralizer algebra of a permutation representation as the adjacency algebra of the cc afforded by the group action. Associated with a weight which is regular on a cc $\mathfrak{X}$ there is a weighted version of the adjacency algebra of $\mathfrak{X}$. The centralizer algebra of a monomial representation is a weighted version of the centralizer algebra of the underlying permutation representation ([7] and section 5 below). This fact made possible a monomial version (presented in full in [7]) of Scott's theory [10] of orbital characters. In addition it provides a method for determining the degrees of the irreducible constituents of monomial characters which was used, for example, in [6] in connection with the Baby Monster. From the point of view of weights in general, the result provides an abundant source of weights which are regular on cc's.

This approach to monomial representations was suggested by the discussion in Taylor's paper [12] of the interpretation of two-graphs as switching classes of graphs and the examples given there of regular two-graphs related to certain doubly transitive monomial representations. There is a natural generalization of Seidel switching under which weights fall into switching classes. Two-graphs can be viewed as examples of switching classes of weights, and the regularity condition for weights generalizes that for two-graphs. In the present paper, which is an update of [5], we review and extend the theory of weights, making the notation and terminology consistent with that of [8], with particular attention to extensions of various aspects of the theory of two-graphs. Taylor's paper [12] and the review paper [11] of Seidel and Taylor are our basic references for two-graphs. The sections of the present paper are as follows:

BULLETIN DE LA SOCIETE MATHEMATIQUE DE BELGIQUE,t.XLII,1990

Regular 3-graphs

- Reworked by Kalmanovich
- Analysis of parameters
- Rank 4 and rank 6 schemes from 3-fold cover of K_n

A NOTE ON REGULAR 3-GRAPHS

D. G. Higman

1. INTRODUCTION

The purpose of this note is to present some elementary considerations which lead to conditions on the parameters of regular 3-graphs. Just 70 sets of feasible parameters of order ≤ 1000 survive these conditions. The first group in the Atlas [2] giving rise to a regular 3-graph via a 2-transitive monomial representation is $U_3(8)$. This example of order 513 is the smallest 3-graph in the family of t-graphs derived from monomial representations of the groups $U_3(q)$ as described in the final section. Two-graphs are the case $t = 2$ and 3-graphs the case $t = 3$ of t-graphs as in [3] to which we refer for background and references. It is our intention to treat extensions of the material of the present note to regular t-graphs for arbitrary t in a later paper in the context of a general discussion of t-graphs. It seems worthwhile to look first at 3-graphs as a step away from two-graphs. (We refer the reader to the survey [5] for background and references on two-graphs.) A step away from two-graphs in a different direction (but still in the context of [3]) are regular weights on strongly regular graphs; these have been studied by A. D. Sankey [4] in the case in which the values are ± 1.

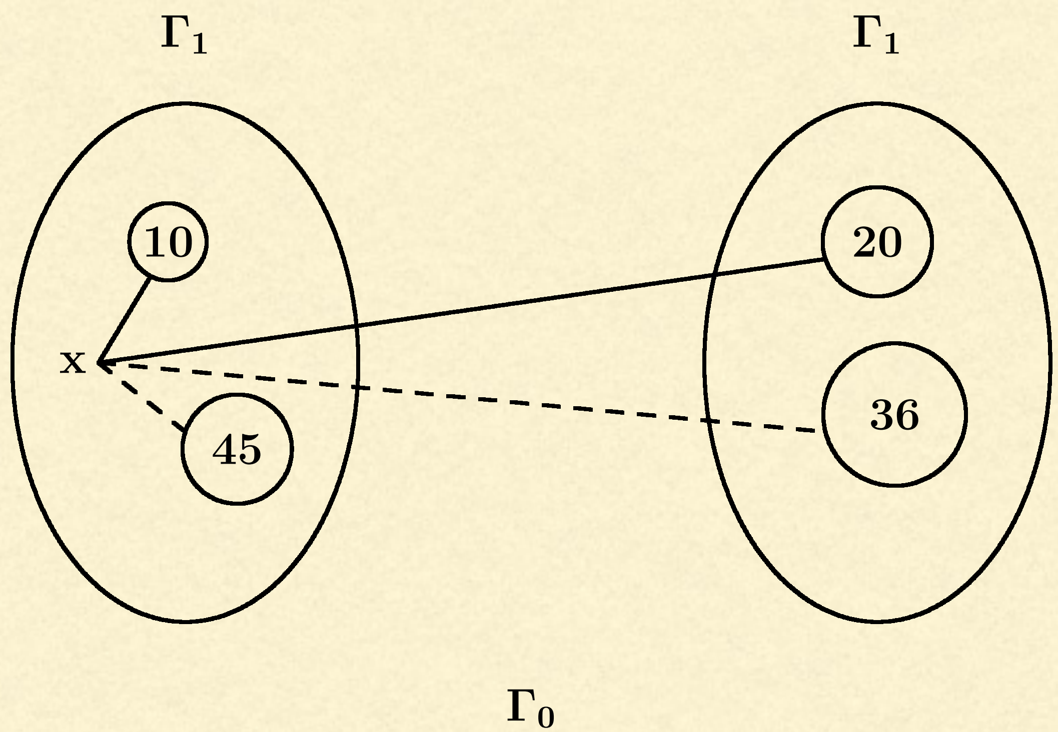
2. 3-GRAPHS AND ASSOCIATION SCHEMES

A t-graph F on a finite ~~on a finite~~ set X as defined in [3] is a function F from X^3 into a cyclic group of order t which is an alternating cocycle such that the values of F generate the cyclic group. Here alternating means that $F(x) = 1$ if any two components of x are equal and $F(y) = F(x)^{-1}$ if y is the result of interchanging two components of x, and cocycle means $\delta F = 1$,

1

A strongly regular decomposition

- srg(112,30,2,10) split into two srg(56,10,0,2)



- decomposition gives rank 5 CC on 112 (symmetric SRD)
- in fact: uniform, cometric Q-antipodal
- (first) example of a hemisystem
- natural (regular) weight on Γ_0 has minimal closure
- weight induces rank 6 cometric cover

More on hemisystems

- Segre ('65) only one known at the time
- Cossidente and Penttila '05: infinite family $H(3,q^2)$ for q odd prime power
- Penttila and Williford '09: new families of cometric schemes

Minimal closure

- Well-known result: regular two-graph from SRG with

$$n = 2(2k - \lambda - \mu)$$

- Use SRG to make regular weight on K_n
- Minimal closure means that the relations determined by the distinct edge weights form a CC; here just the SRG
- SRD example exhibits minimal closure: the weight on Γ_0 would split non-trivial relations in 2, and the rank 5 config is coherent

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