From Transposition Groups to Algebras

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Symmetry vs Regularity
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Finite simple groups

Finite simple groups are the building blocks of all finite groups.

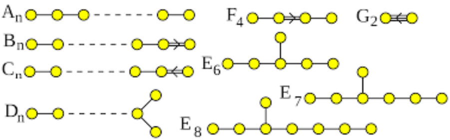
Theorem (CFSG, 1981)

Every finite simple groups is one of the following;

- $\mathbb{Z}_p$, *p a prime number;*
- *an alternating group* $Alt(n)$*;*
- *a group of Lie type;*
- *one of* 26 *sporadic simple groups.*

Lie theory

Groups of Lie type are finite analogues of Lie groups and they arise as groups of automorphisms of Lie algebras. Those are classified by root systems and, eventually, by Dynkin diagrams:



The theory of groups of Lie type was finalized by Steinberg in 1959. Alternating groups are included in this theory, but none of the sporadic groups is.

Transposition groups

Definition

A *transposition group* is a pair (G, D), where G is a group and D a normal generating set of involutions in G.

So:

- if $a \in D$ then $|a| = 2$ and
- $a^g \in D$ for all $g \in G$;
- furthermore, $G = \langle D \rangle$.

Additional properties specify particular classes of transposition groups.

Example

(G, D) is an *n-transposition group* if $|ab| \leq n$ for all $a, b \in D$.

3-transposition groups

3-transposition groups have been classified in the primitive case by Fischer in 1970 and in complete generality by Cuypers and Hall in 1995.

Theorem (Fischer)

A primitive 3-transposition group (G, D) is one of the following:

- $G = Sym(n)$, $D = (1,2)^G$;
- $G = O^{\epsilon}_{2n}(2).2$, *D is the class of transvections;*
- $G = S_{2n}(2)$, *D is the class of transvections;*
- $G = U(n, 2)$, *D is the class of transvections;*
- $G = O^{\epsilon}_{n}(3).2$, *D is a class of reflections;*
- $G = O^{+}_{8}(2) : S_3$ *or* $O^{+}_{8}(3) : S_3$;
- $G = Fi_{22}$, Fi_{23}, *or* Fi_{24}.

Monster

Further study of 4-transposition group led Fischer to the discovery in early 1970s of the Baby Monster B. Finally, the *Monster* M, which is a 6-transposition group, was conjectured to exist in 1973 independently by Fischer and Griess. It was constructed in 1982 by Griess as a group of automorphisms of a non-associative commutative unital algebra of dimension $196,884$ over $\mathbb{R}$, the *Monster algebra* V.

The prime numbers involved in the order of the Monster:

$$|M| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

are very familiar to number theorists. Ogg noticed that they are exactly the fifteen *supersingular primes* arising in the theory of elliptic curves.

Monstrous Moonshine

In 1978 McKay noticed further relation to *modular functions*:

$$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

and

$$\begin{aligned} 196884 &= 1 + 196883, \\ 21493760 &= 1 + 196883 + 21296876, \\ 864299970 &= 2 \cdot 1 + 2 \cdot 196883 + 21296876 + 842609326, \\ &\ldots, \end{aligned}$$

where the numbers on right are the character degrees of the Monster.

Monster VOA

McKay suggested that a natural graded module for M might exist. Conway and Norton did further computations suggesting that the entire graded character of such module should consist entirely of *Hauptmodul* modular functions (and all of them arise).
The conjecture was proved first in 1980 by Atkin, Fong and Smith, but an explicit construction of the module was achieved by Frenkel, Lepowsky and Meurman in 1988 using vertex operators from physics. They further noticed that the module $V^\natural$ carries the structure of a *vertex operator algebra* (VOA), although the proper theory of vertex algebras was only developed by Borcherds in 1992, who finally settled the Moonshine conjecture (Fields medal 1998).
There are many other conjectures tying the Monster VOA $V^\natural$ to many parts of mathematics and to physics, particularly to quantum gravity.

Miyamoto observation

A VOA is a graded algebra $\oplus_{i\geq 0} V_i$ with infinitely many products.
In $V^\natural$, the weight two component V_2 taken with one of the products is the Monster algebra V.
In 1998 Miyamoto observed that special conformal vectors a in V_2 (called *Ising vectors*) lead to involutive automorphisms τ_a of the entire VOA. These are called *Miyamoto involutions*. Hence a nontrivial group of symmetries arises directly from the properties of the algebra elements. The Monster algebra V is full of Ising vectors and the corresponding Miyamoto involutions are 2A-involutions in the Monster group M.
Miyamoto attempted to classify VOAs generated by two such conformal vectors and succeeded in some cases.

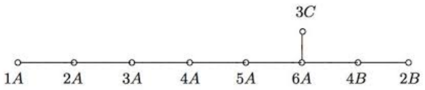
Sakuma's theorem

This work was completed by Sakuma in 2007.

Theorem

There exist exactly eight different VOA generated by two Ising vectors.

All of them are present inside the Monster VOA $V^\natural$.



Majorana algebras

Ivanov noticed that all Sakuma's calculations are done in V_2 using a number of specific properties of this algebra. These became the axioms of *Majorana algebras* introduced by Ivanov in 2009.
They are commutative non-associative algebras A over $\mathbb{R}$ generated by special *Majorana vectors* a that are idempotents (i.e., $a^2 = a$) and

$$A = A_1(a) \oplus A_0(a) \oplus A_{\frac{1}{4}}(a) \oplus A_{\frac{1}{32}}(a),$$

where $A_\lambda(a) = \{u \in A \mid au = \lambda u\}$.
In other words, the *adjoint map* $ad_a : A \to A$ (defined by $u \mapsto au$) has eigenvalues in the set $\{1, 0, \frac{1}{4}, \frac{1}{32}\}$ and it is semisimple (has a basis consisting of eigenvectors).

Majorana fusion rules

Multiplication of eigenvectors of ad_a is governed by the following *fusion rules*:

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1+0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1+0+\frac{1}{4}$

There are several further axioms: in particular, *primitivity* requires that $A_1(a) = \langle a \rangle$, and there must be a (positive definite) inner product on A that associates with the algebra product:

$$(u, vw) = (uv, w)$$

for all $u, v, w \in A$.

Majorana involution

Set $A_+ = A_1(a) \oplus A_0(a) \oplus A_{\frac{1}{4}}(a)$ and $A_- = A_{\frac{1}{32}}(a)$. Then the linear map $\tau_a : A \to A$ that acts as identity on A_+ and as minus identity on A_- is an automorphism of A, the *Majorana involution*. Hence every Majorana algebra has a substantial group of symmetries. One may ask: which groups arise in this way from Majorana algebras?

Early successes

In [IPSS2010] Sakuma's theorem was reproved in the context of Majorana theory. That is, all 2-generated Majorana algebras are known and they are exactly the same eight Sakuma algebras that arise within the Monster algebra V. The implication of this is that every group arising from a Majorana algebra is a 6-transposition group.

This also allowed to start computing Majorana algebras for concrete groups in terms of *shapes*, specifying which Sakuma algebras arise within the Majorana algebra. In the 2010 paper algebras for S_4 for four shapes were determined. Different subsets of the above four authors worked on Majorana algebras for various small simple groups: A_5, $L_3(2)$, A_6, $L_2(11)$,... In particular, Seress had great success computing 2-*closed* algebras in computer algebra system GAP.

Largest Majorana algebra?

For a while all newly constructed algebras were subalgebras of the Monster algebra V. So the (naive) conjecture was that the Monster algebra was the unique largest Majorana algebra.

In three Sakuma algebras, $2A$, $2B$, and $3C$, one of the eigenspaces $A_{\frac{1}{4}}(a)$ or $A_{\frac{1}{32}}(a)$ is trivial. Consequently, the fusion rules simplify to:

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1+0$

where $\eta = \frac{1}{4}$ or $\frac{1}{32}$.

Furthermore, the group acting on such an algebra must be a group of 3-transpositions.

Matsuo algebras

Consider a field $\mathbb{F}$ of characteristic not equal to 2 and an arbitrary group (G, D) of 3-transpositions.
Take D as basis of an algebra A over $\mathbb{F}$ and define multiplication as:

$$c \cdot d = \begin{cases} c, & \text{if } |cd| = 1 \text{ (hence } c = d) \\ 0, & \text{if } |cd| = 2 \text{ (hence } cd = dc) \\ \frac{\eta}{2}(c + d - e), & \text{if } |cd| = 3 \text{ (and hence } c^d = d^c =: e) \end{cases}$$

These are called *Matsuo algebras*. The basis vectors $d \in D$ are idempotents and they always satisfy the above fusion rules, for any η. This examples shows the need to break with the strict confines of Majorana theory.

Fusion rules

Definition

Let $\mathbb{F}$ be a field and let $\mathcal{F} \subseteq \mathbb{F}$. A *fusion table* on $\mathcal{F}$ is a binary operation $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$.

By abuse of notation, we denote the fusion table by the same symbol $\mathcal{F}$, as the underlying set. We also talk, as above, about *fusion rules* $\mathcal{F}$ meaning individual entries of $\mathcal{F}$.
We have seen two examples of fusion rules. The first one, 4×4, will be denoted by $\mathcal{H}(\alpha, \beta)$, and we will allow arbitrary α and β, not just $\frac{1}{4}$ and $\frac{1}{32}$. The second one, 3×3, will be denoted by $\mathcal{J}(\eta)$.

Axis

Let $\mathcal{F} \subset \mathbb{F}$ be fusion rules and A be a commutative non-associative algebra over $\mathbb{F}$. For $X \subseteq \mathcal{F}$ and $a \in A$, we write $A_X(a)$ for $\oplus_{\lambda \in X} A_\lambda(a)$.

Definition

An idempotent $a \in A$ is an ($\mathcal{F}$-)axis if

- $A = A_{\mathcal{F}}(a)$; and
- $A_\lambda(a)A_\mu(a) \subseteq A_{\lambda * \mu}(a)$ for all $\lambda, \mu \in \mathcal{F}$.

This assumes that $1 \in \mathcal{F}$, as $\mathrm{ad}_a(a) = aa = a = 1a$.
The axis a is primitive if $A_1(a) = \langle a \rangle$.

Axial algebra

Definition

A commutative non-associative algebra A is a (primitive) ($\mathcal{F}$-)axial algebra if it is generated by (primitive) ($\mathcal{F}$-)axes.

We may assume:

- $\mathcal{F}$ is commutative: $\lambda * \mu = \mu * \lambda$ for all $\lambda, \mu \in \mathcal{F}$.
- In the primitive case, $1 * \lambda = \{\lambda\}$, if $\lambda \neq 0$, and $1 * 0 = \emptyset$.

Graded fusion rules

For fusion rules $\mathcal{F}$ and a group T, a T-grading on $\mathcal{F}$ is a partition $\mathcal{F} = \cup_{t \in T} \mathcal{F}_t$ such that

- $\mathcal{F}_s * \mathcal{F}_t \subseteq \mathcal{F}_{st}$ for $s, t \in T$
  (where $\mathcal{F}_s * \mathcal{F}_t := \{\lambda * \mu \mid \lambda \in \mathcal{F}_s, \mu \in \mathcal{F}_t\}$).

Useful only, please:

- $T = \langle t \in T \mid \mathcal{F}_t \neq \emptyset \rangle$.

Theorem

Every $\mathcal{F}$ admits a unique finest grading for a suitable largest T.

Axial subgroup

Let T^* denote the group of linear characters of T.
For a an axis and $\xi \in T^*$, define $\tau_a(\xi) : A \to A$ via

$$\tau_a(\xi)|_{A_\lambda(a)} := \xi(t) Id,$$

where t is defined by $\lambda \in \mathcal{F}_t$.
Set

- $Axial(a) := \{\tau_a(\xi) \mid \xi \in T^*\}$;
- $G^\circ := \langle Axial(a) \mid a \text{ an axis} \rangle$.

Then $Axial(a)$ and G° are subgroups of $Aut(A)$.

Algebras of Jordan type

An algebra of Jordan type η is a primitive $\mathcal{J}(\eta)$-axial algebra, where $\mathcal{J}(\eta) = \{1, 0, \eta\} \subset \mathbb{F}$.

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1+0$

These fusion rules are C_2-graded, and so for every axis A, there is a corresponding axial subgroup of order 2, $\langle \tau_a \rangle$.
Matsuo algebras are algebras of Jordan type and so are, but just for $\eta = \frac{1}{2}$, Jordan algebras generated by primitive idempotents.

Jordan algebras

Every associative algebra taken with the anti-commutator product $u \circ v = uv + vu$ becomes a *Jordan algebra*, that is, a commutative non-associative algebra A satisfying the Jordan identity:

$$(u^2 v)u = u^2(vu).$$

If $a \in A$ is an idempotent then A has the *Peirce decomposition*

$$A = A_1(a) \oplus A_0(a) \oplus A_{\frac{1}{2}}(a)$$

and the multiplication of eigenvectors is exactly as in fusion rules $\mathcal{J}(\frac{1}{2})$.

Jordan algebras exist for all classical groups and some exceptional groups. Hence within the paradigm of axial algebras we bring together most groups of Lie type and most sporadic groups!

Sakuma type theorem

For algebras of Jordan type we know all 2-generated algebras.

Theorem (HRS2015-1)

Either $A = \langle\langle a, b\rangle\rangle$ is $2B \cong \mathbb{F}^2$ (where $ab = 0$) or it is spanned by a, b, and $\sigma = ab - \eta a - \eta b$. The multiplication is given by:

$\cdot$	a	b	σ
a	a	$\frac{1}{2}a + \frac{1}{2}b + \sigma$	πa
b	$\frac{1}{2}a + \frac{1}{2}b + \sigma$	b	πb
σ	πa	πb	$\pi \sigma$

where $\pi = \eta - \phi\eta - \phi$. Furthermore, if $\eta \neq \frac{1}{2}$ then $\phi = \frac{\eta}{2}$.

Hence, generically, for most η, there are only two algebras, and for the exceptional value $\eta = \frac{1}{2}$, the 2-generated algebra A is identified by the value of ϕ, which can be arbitrary.

Generic case

As a consequence, we can classify all Jordan type algebras of type $\eta \neq \frac{1}{2}$.

Theorem (HRS2015-1+HSS2017-1)

If $\eta \neq \frac{1}{2}$ then every algebra of Jordan type is a Matsuo algebra or a quotient.

The case $\eta = \frac{1}{2}$ is more difficult and it is currently still open. However, we can say a bit more about this case.

Frobenius form

Often A admits a bilinear form satisfying

- $(u, vw) = (uv, w)$ for all $u, v, w \in A$;
- $(a, a) = 1$ for every axis a.

For example, a Majorana algebra by definition admits a positive definite Frobenius form.

Theorem (HSS2017-2)

Every algebra of Jordan type admits a Frobenius form.

Generalizing Sakuma

Ideally, we want the original Sakuma's statement without any extra assumptions, just in terms of the fusion rules.

Theorem (HRS2015-2)

There exists a universal k-generated axial algebra with Monster fusion rules and a Frobenius form.

This algebra is defined over a ring!

Corollary

The list of 2-generated algebras over a field of a sufficiently large characteristic with Monster fusion rules and a Frobenius form is the same as in Sakuma's theorem.

Rehren's attempt

Rehren attempted to prove a Sakuma theorem for arbitrary fusion rules $\mathcal{H}(\alpha, \beta)$ and without assuming the existence of a Frobenius form..

Theorem (R2015)

Assuming that $\alpha \neq 2\beta$ and $\alpha \neq 4\beta$, a 2-generated algebras with fusion rules $\mathcal{H}(\alpha, \beta)$ has dimension at most 8.

In December 2017, Franchi, Mainardis and Shpectorov re-checked Rehren's computations and obtained slightly different formulas.

Theorem (FMS, in preparation)

Over a field of characteristic zero, a 2-generated algebra with the Monster fusion rules is necessarily one of the eight Sakuma algebras.

Double axes

Let A be an algebra of Jordan type η (say, a Matsuo algebra). If a and b are two *orthogonal* axes, that is, $ab = 0$, then $x = a + b$ is an idempotent called a *double axis*.

Theorem (JS+3A, in preparation)

Double axes obey fusion rules $\mathcal{H}(2\eta, \eta)$.

$*$	1	0	2η	η
1	1		2η	η
0		0	2η	η
2η	2η	2η	$1+0$	η
η	η	η	η	$1+0+2\eta$

Double axes are not primitive in A, however we managed to build many primitive algebras generated by double axes, including an infinite series of dimension n^2.

Thank you for listening!!