Finding a maximal subgroup of minimal index in polynomial time

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Motivation

Graph = a finite undirected graph without loops or multiple edges.

We say that a representation of a group G on a graph Γ is a homomorphism from G to $Aut(\Gamma)$.

Notice that there always exists a trivial representation sending all elements of G to id_Γ.

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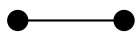
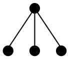
Example

Sym_4 has a nontrivial representation on 

Alt_4 has no nontrivial reprs. on , but has one on 

Group representability problem

Given a group G by its multiplication table and a graph Γ by adjacency matrix, determine whether there exists a nontrivial representation of G on Γ.

Theorem (Dutta and Kurur, 2009)

Graph isomorphism problem is reducible to the group representability problem for groups of the form C_p, where p is a prime.

Tree isomorphism can be decided in linear time.

Theorem (Dutta and Kurur, 2009)

The group representability problem on trees is equivalent to the problem of testing, given an integer n and a group G, whether there exists a nontrivial homomorphism from G to Sym_n.

The latter problem is called permutation representability problem and it is equivalent to finding the smallest integer n, such that

$$\exists \phi : G \to Sym_n,\ \phi \not\equiv 1$$

Denote by $\kappa(G)$ the smallest possible degree of a nontrivial permutation representation of G.

Such a representation is primitive.

In other words:

$$\kappa(G) = \min\{|G : M| \mid M \text{ is maximal in } G\}.$$

To recap:

Group representability on trees
(Does G act on a tree?)
$\Updownarrow$
Permutation representability problem
(Does G act on d points?)
$\Updownarrow$
Computing $\kappa(G)$
(What is the smallest $|G : M|$ for $M < G$?)

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Theorem

$\kappa(G)$ can be computed in time polynomial in $|G|$.

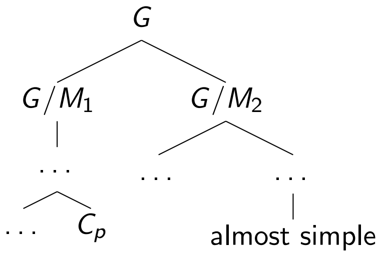
Corollary

Group representability on trees can be decided in polynomial time.

The algorithm substantially relies on CFSG.

Algorithm outline

On each step we select two minimal normal subgroups.



Theorem (Burness, Liebeck, Shalev, 2013)

Maximal subgroups of almost simple groups are 6-generated.

Height of the tree is $\leq \log_2 |G| \Rightarrow$ running time $\sim |G|^{11}$.

Permutation groups

There are two main obstacles in applying this algorithm to permutation groups:

- Traversing the tree of quotients
  (having to look at $2^{\log_2 |G|} = |G|$ in the worst case)
- Generating maximal subgroups in the almost simple case
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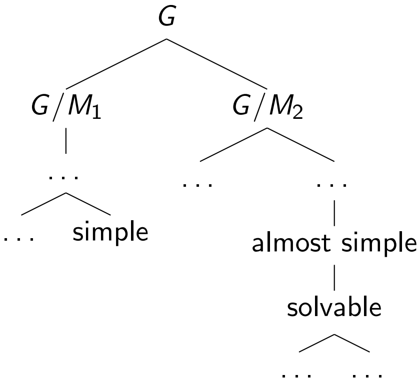
Both can be overcome.

Lemma

Let $S < G \leq Aut(S)$, where S is a nonabelian simple group, and pick $H \leq G$ with $|G : H| = \kappa(G)$. Then $S \leq H$.

The proof relies on Schreier's hypothesis.



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Theorem

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Corollary

Let G be a permutation group. Then $\kappa(G)$ can be computed in polynomial time.

Proof.

Find all simple quotients G/N and compute $\kappa(G/N)$ by using the description of minimal permutation representations of simple groups. □

Some questions

- Are there any applications besides group representability on trees?
- What about group representability on other classes of graphs with controllable automorphism group?
(planar graphs, for example)

A final remark

Recall the main result:

Theorem

Let H be a subgroup of minimal index in G. Then $G/core_G(H)$ is a simple group.

Compare with these:

Proposition 1

Let p be a smallest prime dividing $|G|$. If $H \leq G$ and $|G:H| = p$, then $H \trianglelefteq G$.

Proposition 2

Let $f = \min\{\chi(1) \mid \chi \in Irr(G), \chi(1) > 1\}$. If $H \leq G$ and $|G:H| \leq f$, then $H \trianglelefteq G$.