Discrete version of Fuglede's conjecture and Pompeiu problem

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3. July 2018.
Symmetry vs. Regularity

Our source of interest, the Pompeiu problem

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- Unit disc is not a Pompeiu set.
- Brown, Schreiber, Taylor (1973): Nonempty polygons (domains having a corner) have the Pompeiu property.
- Ramm (2017): Every domain having a smooth boundary is Pompeiu except the unit ball.

Fuglede's conjecture

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Fuglede proved that in some special cases (e.g. T is a lattice, Ω is open) the previous two properties are equivalent. He conjectured (1974) that the two properties are equivalent for every domain.

Discrete version of the problems

- Pompieu: Let G be a finite (countable) abelian group, S a (finite) subset of G. Does there exists a nontrivial function from G to $\mathbb{C}$ with $\sum_{s\in S} f(x+s) = 0$ for every $x \in G$?

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- Fuglede: S tiles G if and only if there exists $T \subseteq G$ such that every element of G can be uniquely written as $s + t$ ($s \in S$, $t \in T$). The exponential functions are the irreducible representations of G.
- Orthogonality comes from the usual scalar product of L^2-functions restricted to S.

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His reformulation of the spectral property for finite abelian groups:
Let S be a spectral set.
H is a $k \times k$ submatrix of the character table of the finite abelian group. The rows of H are indexed by the elements of the spectral set S the columns are indexed by the elements of the spectrum. Then the resulting matrix is a complex Hadamard matrix.

Scketch of the proof

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Take S be a basis of $\mathbb{Z}_2^{12}$. Then the prescribed Hadamard matrix can be achived as the submatrix of the character table so S is a spectral set, while 12 does not divide 2^{12} so S does not tile $\mathbb{Z}_2^{12}$.

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Tao's original idea gives the same for $\mathbb{Z}_3^5$ using complex Hadamard matrices.
He lifted up this counterexample for the continuous case.

Elementary abelian groups

- The other direction was disproved using by Kolountzakis and Matolcsi (2006).
- Matolcsi (2005): Counterexample for the Spectral-Tile direction for elementary abelian groups of rank 4.
- Farkas, Matolcsi and Móra (2006): The continuous version of the original conjecture does not hold in $\mathbb{R}^3$.
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  We managed to reprove this result using simple methods.

Reformulation of Pompeiu problem, Connection of the problems.

Recall: Let G be a finite (countable group) abelian group, S a subset of G. Does there exists a nontrivial function from G to $\mathbb{C}$ with $\sum_{s \in S} f(x+s) = 0$ for every $x \in G$?

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Immediate from this that S is a non-Pompeiu set if and only if the adjacency matrix of $Cay(G,S)$ is singular. Its eigenvalues (G is abelian) are of the form $\sum_{s \in S} \chi(s)$, where $\chi \in Irr(G)$.

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However, the connection of the problems is transparent since orthogonality of characters restricted to S means:

$$0 = \sum_{s\in S} \chi_1(s)\bar{\chi_2}(s) = \sum_{s\in S} (\chi_1\bar{\chi_2})(s).$$

$\chi_1\bar{\chi_2}$ is also a character of G.

1 dimensional case of Fuglede's conjecture

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If S is a finite subset of $\mathbb{Z}$, which is also a tile and T is a tiling complement, then T is periodic ($T+n=T$ for some $n \in \mathbb{N}$). Thus we are left to understand the tiles of finite cyclic groups.

Coven-Meyerowitz conjecture 1

- The natural version of the conjecture:
  Let n be square free. Then $S \subset \mathbb{Z}_n$ tiles $\mathbb{Z}_n$ if and only if S is a complete set of coset representatives for some subgroup of $\mathbb{Z}_n$.

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Definition

The **mask polynomial** *(an object in $\mathbb{Q}[x]/(x^n - 1)$) of a set $S \subseteq \mathbb{Z}_n$ is $S(x) = \sum_{s \in S} x^s$.*

H_S is the set of prime powers r^a dividing n such that $\Phi_{r^a}(x) \mid S(x)$.

(T1) $|S| = S(1) = \prod_{d \in H_S} \Phi_d(1)$.

(T2) For pairwise relative prime elements s_i of H_S, $\Phi_{\prod s_i} \mid S(x)$.

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The generalization of Coven-Meyerowitz conjecture is that the converse of the statement of the theorem also holds.
There is an argument on Tao's blog by (Izabella Laba) claiming that Conjecture 1 holds but an independent proof was uploaded recently to Arxiv by Ruxi Shi (29. 05. 2018.).

Our results

Previous results on cyclic groups:

- Fuglede's conjecture holds for $\mathbb{Z}_p$ (trivial).
- Fuglede's conjecture holds for $\mathbb{Z}_{p^m}$ (Laba).
- Kolountzakis-Malikiosis: Fuglede's conjecture holds for $\mathbb{Z}_{p^m q}$.

Theorem

Let p, q and r be different primes.

1. *Fuglede's conjecture holds for $\mathbb{Z}_{p^m q^2}$.*
2. *Fuglede's conjecture holds for $\mathbb{Z}_{pqr}$.*

The latter result was independently proved by Ruxi Shi (2018).

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Let S be a spectral set with Λ the spectrum.

1. $0 \in S$, $0 \in \Lambda$ and each of S and Λ generates $\mathbb{Z}_{p^n q^2}$.
   Moreover $S(\xi_{p^m q^l}) = 0$
2. Both S can be written as the disjoint union of $\mathbb{Z}_p$-cosets and $\mathbb{Z}_q$-cosets and this holds the intersections of S with each $\mathbb{Z}_{pq}$-cosets as well.
   This comes from the description of non-Pompeiu sets.
3. There is a $\mathbb{Z}_{pq}$-coset which intersects S and its complement. Further the intersection is the union of $\mathbb{Z}_p$-cosets. The same holds for another $\mathbb{Z}_{pq}$-coset with $\mathbb{Z}_q$-cosets as well.

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5. Spectral implies (T1) and (T2) implies Tile. Works for $\mathbb{Z}_{p^m q^2}$.

Method for pqr

- One has to handle small sets $|S| \leq 5$ separately using the theory of complex Hadamard matrices.
- The case when $S(\xi_{pqr}) \neq 0$ requires some finite geometry argument.
- If $S(\xi_{pqr}) = 0$, then we use a generalization of 2. from the prevoius page:
  S is the weighted sum of cosets of $\mathbb{Z}_p$, $\mathbb{Z}_q$ and $\mathbb{Z}_r$ with rational weights.

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  S is the weighted sum of cosets of $\mathbb{Z}_p$, $\mathbb{Z}_q$ and $\mathbb{Z}_r$ with rational weights.
  If the weights are nonnegative, then the argument is similar to the one for $p^m q^2$.
  If some of the weights are negative, then S is huge:
  $|S| \geq (p-1)(q-1) + r - 1$, where $r > p, q$.