Graph isomorphism and Schurian Polynomial Approximation Schemes

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Schurian polynomial approximation schemes (SPAS), Ponomarenko et. al., 1999

Definition (SPAS)

For each $i \in \mathbb{N}$ we let X_i be some map from the set of coherent algebras over V to itself. We say that $X = \{X_i \mid i \in \mathbb{N}\}$ is a SPAS if for every coherent algebra W the following hold:

- $X_1(W) = W \leq X_2(W) \leq \ldots \leq X_n(W) = \ldots = Sch(W)$.
- For all $i \leq j$ and coherent algebras W, $X_i(X_j(W)) = X_j(W)$.
- The standard basis for $X_i(W)$ is computable in time $n^{O(i)}$ with $n = |V|$.

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Definition (Equivalence of SPAS)

We say that X and Y are equivalent SPAS if there are maps $\mu, \nu : \mathbb{N} \to \mathbb{N}$ such that for any coherent algebra W $X_{\nu(k)}(W) \leq Y_k(W) \leq X_{\mu(k)}(W)$.

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For any coherent algebra W

Ponomarenko, Evdokimov, 1999

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$$WL_{k-r}(W) \leq WL_{k,r}(W) \leq WL_{k+r-1}(W)$$

where $WL_{k,r}$ arises from labelling k-tuples with multisets of maps $t : [k]^{(r)} \to range(f)$ as opposed to $t : [k] \to range(f)$ as in WL_k.

A hypothesis a la Church-Turing?

Hypothesis

Any reasonable Schurian polynomial approximation scheme is equivalent to WL.

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FALSE

Invertible map refinements

χ-matrix

For each $t : [k]^{(2)} \to range(f)$, $\vec{v} \in V^k$, define the matrix $\chi^t_{\vec{v}}$ labelled by V to have (u, v) entry equal to 1 if $t(\tau) = f(\vec{v}\langle\tau, (u, v)\rangle)$ for all $\tau \in [k]^{(2)}$.

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IM_k stability condition

$\vec{u} \equiv^{k,\mathbb{F}}_{\mathrm{IM}} \vec{v}$ if there is a matrix $S \in GL_V(\mathbb{F}$ such that for every $t : [k]^{(2)} \to range(f)$ we have that $S\chi^t_{\vec{v}}S^{-1} = \chi^t_{\vec{u}}$.

Theorem

Consider IM over $\mathbb{C}$. Then for any coherent algebra W

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Consider IM over $\mathbb{Z}_p$. Then for any coherent algebra W

$$WL_{k-2}(W) \leq IM_k(W).$$

Constructions

Theorem (Cai-Fürer-Immerman, 1992)

For every k there are graphs G such that $\chi(G)$ and $\tilde{\chi}(G)$ are not isomorphic but $\chi(G) \equiv_{\mathrm{WL}} \tilde{\chi}(G)$.

Constructions

Theorem (Cai-Fürer-Immerman, 1992)

For every k there are graphs G such that $\chi(G)$ and $\tilde{\chi}(G)$ are not isomorphic but $\chi(G) \equiv_{\mathrm{WL}} \tilde{\chi}(G)$.

Theorem (Holm, 2012)

Let p, q be distinct primes. Then for each k there are graphs G such that $\mathcal{H}^k(G) \equiv_{\mathrm{IM}} {}_k^p\tilde{\mathcal{H}}^k(G)$ but are distinguished by IM_k^q.

Thus we cannot expect relations of the form $IM_k^p(W) \leq WL_{f(k)}(W)$ or $IM_k^p(W) \leq IM_{f(k)}^q(W)$.

Open questions

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Too many