On multiset reconstruction index of 2-nilpotent graph groups

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Symmetry vs Regularity

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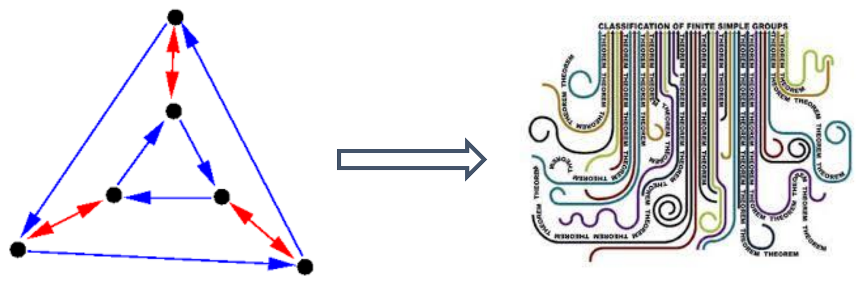
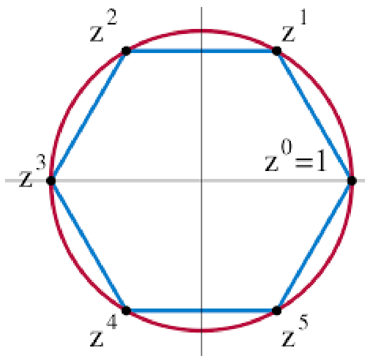
Graphs and groups (history)



- Finite algebraic structures are reducible to graphs in polynomial time
- Inverse isomorphism-preserving reduction exists for some structures
  - Rings (N. Kayal, N. Saxena 2006)
  - Infinite-dimensional Lie algebras (K.H. Kim, L. Makar-Limanov, J. Neggers, F.W. Roush 1980)
  - Infinite groups (C. Droms 1987)
  - Finite p-groups (P. Beletskii 1985)

Graphs to 2-nilpotent finite p-groups



- Results of (Lipyanski, Vanetik 2015)
  - Graphs are reducible to class 2 nilpotent Lie algebras
  - Graphs are reducible to 2-nilpotent finite p-groups
  - In both cases, we have P-Borel reducibility (poly-time, isomorphism preserving functor)
- Good things
  - Isomorphism problem for graphs is wild (contains pair-of-matrices problem)
  - It is actually superwild (is not contained in the pair-of-matrices problem)
- Bad things
  - One cannot reduce graphs to a tame category, say Abelian groups

Graphs to 2-nilpotent finite p-groups

- Graph group $G(\Gamma) = M_n / J$ where
  - M_n is a free group generated by *T* is group variety determined by
    $$x^p = 1, [[x, y], z] = 1$$
  - J is a (normal) subgroup generated by
    $$\{v_i \wedge v_j \mid \{v_i, v_j\} \in E\}$$
- $G(\Gamma_1) \cong G(\Gamma_2) \Leftrightarrow \Gamma_1 \cong \Gamma_2$ (Lipyanski, Vanetik 2015)
- Proof uses Lazard correspondence (pls ask Ruvim!)

Reconstruction of groups

Problem: reconstruct action of group G on collection X from its action on size k sub-collections $X^{\{k\}}$ of X

Reconstruction index = minimal k for which action of G can be reconstructed for all X

Depending on group action and type of X, define:

- Orbit reconstruction
- Set reconstruction
- Multiset reconstruction

Set reconstruction number

Some results (a non-survey):

- Alon, Caro, Krasikov and Roditty (1989) showed that $r_{set}(Z_n) < \log_2 n$
- Radcliffes and Scott (1998) showed that $r_{set}(Z_n) < 9 \ln n$
- Pebody (2004) proved that $r_{set}(Z_n)<=6$.
- Pebody (2007) proved that the 3-deck determines all nonnegative rational valued functions up to translation on Z_n iff n is a power of an odd prime or the product of two odd primes.

Multiset reconstruction number - survey

- (Radcliffe, Scott 2006): For a group action

$$G \to X \quad r_N(G \to X) \le r_N(G) \;\; (r_N(G)\,\text{stands for left-regular action})$$

- (Radcliffe, Scott 2006): For $G \to X$ and $\mathrm{N} \lhd \mathrm{G}$

$$r_N(G \to X) \le r_N(N \to X)\, r_N(G/N)$$

- (Radcliffe, Scott 2006): Let *G* and *H* be groups. Then

$$r_N(G \times H) \le r_N(G)\, r_N(H)$$

- (Pebody, 2004): $r_N(G) \le 6$

  for Abelian groups of the form $\mathbf{Z}_p \oplus \ldots \oplus \mathbf{Z}_p$

Reconstructibility of 2-nilpotent graph p-groups

Theorem (V). $r_N(G_\Gamma) \leq 36$ for any 2-nilpotent graph p-group G_Γ.

Proof sketch:

$$I = \left\langle v_i \wedge v_j \mid (v_i, v_j) \notin E \right\rangle \triangleleft G_\Gamma$$

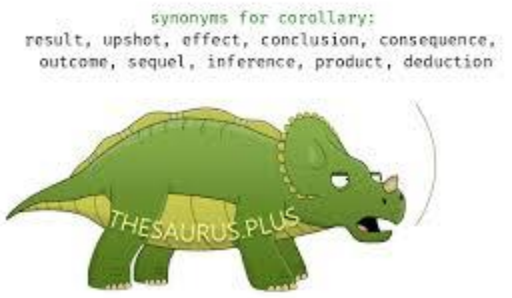
Both I and G_Γ / I are abelian of form $\mathrm{Z}_\mathrm{p}^\mathrm{n}$ for odd p.

In fact, $G_\Gamma / I \cong G_{K_n}$.

Thus $r_N(G_\Gamma) \leq r_N(I) r_N(G_\Gamma / I)$ and $r_N(G_\Gamma \to X) \leq r_N(I) r_N(G_\Gamma / I)$.

Then $r_N(G_\Gamma \to X) \leq r_N(G_\Gamma) \leq 36$.

Corollaries



1. As sets are a subclass of multisets, trivially
2. For *Aut(G)* let us define action $r_{set}(G_\Gamma) \le r_N(G_\Gamma) \le 36.$

$$\mathrm{G} \to \mathrm{Aut(G)} : (G, Aut(G)) \to Aut(G)$$

$$\text{by } (g, \varphi) : G \to G \text{ where } (g, \varphi)(x) = g\varphi(x)g^{-1}$$

Then $\mathrm{r_N}(\mathrm{G} \to \mathrm{Aut(G)}) \le 36$

3. Similarly, for *Out(G)* and this action we have $\mathrm{r_N}(\mathrm{G} \to \mathrm{Out(G)}) \le 36$

But: all automorphisms of G that translate to graph automorphisms are outer, and it's the regular actions of Out(G) that can tell us something about graph reconstructibility…

Thank you!

Questions?