Constructing Majorana representations

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- If $t, s \in 2A$ then ts is of order at most 6 and belongs to one of nine conjugacy classes:

$$1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A.$$

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- The 2A-axes generate the Griess algebra i.e. $V_{\mathbb{M}} = \langle\langle \psi(t) \, : \, t \in 2A \rangle\rangle$.
- If $t, s \in 2A$ then the algebra $\langle\langle \psi(t), \psi(s) \rangle\rangle$ is called a dihedral subalgebra of $V_{\mathbb{M}}$ and has one of nine isomorphism types, depending on the conjugacy class of ts.

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- In particular, we have $\mathrm{Aut}(V^\natural) = \mathbb{M}$ and $V_2^\natural \cong V_{\mathbb{M}}$.

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Theorem (Sakuma 2007)

If V_2 is a generalised Griess algebra and $a_0, a_1 \in V_2$ are Ising vectors then the subalgebra $\langle\langle a_0, a_1 \rangle\rangle$ is isomorphic to one of the nine dihedral subalgebras of the Griess algebra.

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Sakuma's Theorem (Ivanov, Pasechnik, Seress, Shpectorov 2010)

Any Majorana algebra generated by two Majorana axes is isomorphic to a dihedral subalgebra of the Griess algebra.

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such that φ and ψ interact in a "sensible" way.

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Pfeiffer and W. 2018 - reimplementation and extension of Seress's methods (in GAP).

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Otherwise, a `false` value (i.e. no representation exists) or a `fail` value (i.e. algorithm has been unable to determine existence or non-existence of a representation).

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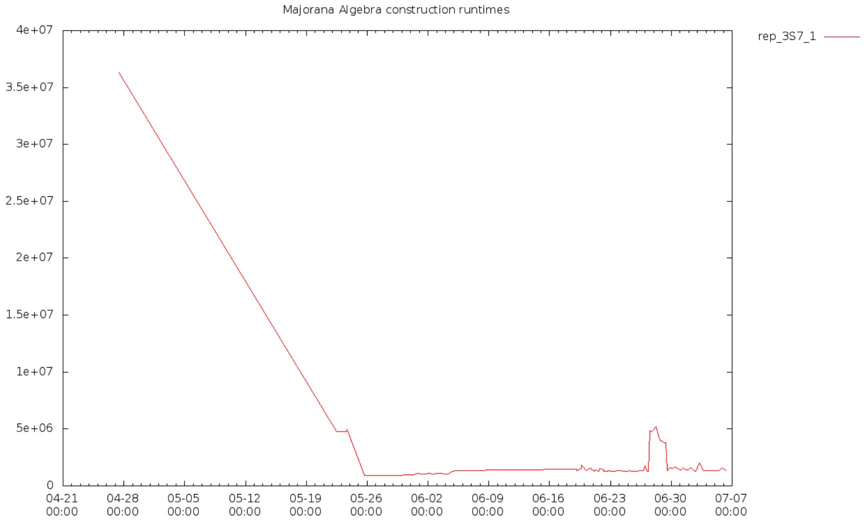
Step 3 - rinse and repeat. Loop over steps 1 and 2 until all products are found.

Performance $G = 3.S_7$, $|T| = 168$, $\dim V = 254$.



Find our work at

- github.com/mwhybrow92/majoranaalgebras
- github.com/gap-packages/Char0Gauss
- arxiv.org/abs/1803.10723