Primitive coherent configurations with very many automorphisms

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Overview

1. Background and motivation
2. Classification of PCCs with very many automorphisms
3. Structure theory for PCCs

Coherent configurations

Definition

Coherent configurations (CCs) **stable colorings under W-L**

Coherent configurations

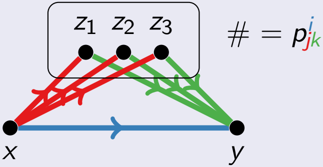
Definition

CC $\mathfrak{X}$ on set Ω is coloring $c : \Omega \times \Omega \to \{\text{colors}\}$ (edge-colored complete digraph with loops) s.t.

(1)

(2)

(3)

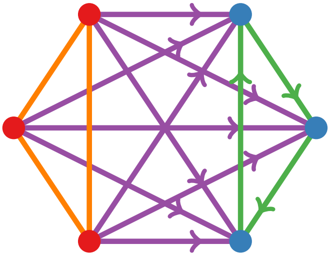


Coherent configurations

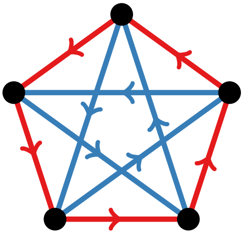
Definition

CC is **homogeneous** if

(4) all vertices (loops) have same color



inhomogeneous



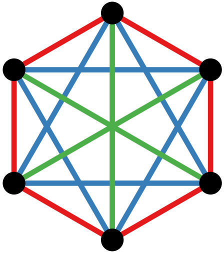
homogeneous

Coherent configurations

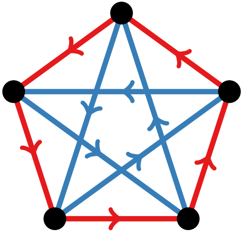
Definition

Homogeneous CC $\mathfrak{X}$ is **primitive (PCC)** if

(5) each constituent digraph $\mathfrak{X}_i = (\Omega, c^{-1}(i))$ is connected

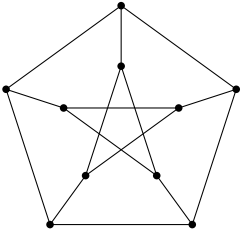


imprimitive



primitive

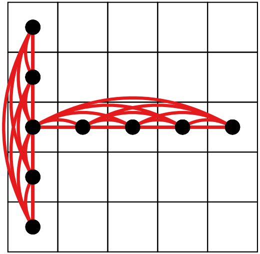
Johnson and Hamming schemes



$J(m, k)$:

Johnson scheme on $\binom{[m]}{k}$

$c(A, B) = |A \cap B|$



$H(m, d)$:

Hamming scheme on $[m]^d$

$c(w_1, w_2) =$ Hamming distance

Schurian CCs

Group action $G \curvearrowright \Omega$ gives CC $\mathfrak{X}(G)$ on Ω:

$$c(u, v) = c(x, y) \text{ iff } (\exists g \in G)(u^g, v^g) = (x, y)$$

- $J(m, k) = \mathfrak{X}(S_m^{(k)})$
- $H(m, d) = \mathfrak{X}(S_m \wr S_d)$

The asymptotic perspective

Interested in **behavior of infinite families**
as number of vertices $n \to \infty$

E.g.,

- $|\operatorname{Aut}(J(m,2))| = \exp(\widetilde{O}(n^{1/2}))$
- $|\operatorname{Aut}(H(m,2))| = \exp(\widetilde{O}(n^{1/2}))$

Symmetry vs. regularity

Symmetry - properties of automorphism group
Regularity - expressed in numerical parameters

All PCCs quite regular...

...but few PCCs are (very) highly symmetric!

Babai's conjecture

Conjecture (Babai)

$(\forall \varepsilon > 0)(\exists n_0)$
every PCC with $n \geq n_0$ *verts. and* $\geq \exp(n^{\varepsilon})$ *automorphisms has primitive automorphism group*

Cameron schemes

Theorem (Cameron 1981 via CFSG)

Every primitive permutation group Γ *has form*

$$(A_m^{(k)})^d \leq \Gamma \leq S_m^{(k)} \wr S_d$$

or has order $n^{O(\log n)}$ *where* n *is degree*

Cameron schemes

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Corollary (Under Babai's conjecture)

Every PCC with $\exp(n^{\varepsilon})$ automorphisms is Schurian $\mathfrak{X}(\Gamma)$ with Γ as above

PCCs with very many automorphisms

Theorem (Babai, 1981)

PCCs other than trivial (K_n) have $\leq \exp(\widetilde{O}(n^{1/2}))$ *automorphisms*

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Theorem (Sun–W 2015)

PCCs other than trivial, $J(m,2)$*, and* $H(m,2)$ *have* $\leq \exp(\widetilde{O}(n^{1/3}))$ *automorphisms*

Classification of primitive permutation groups

Corollary (Sun–W 2015)

Primitive permutation groups of degree n
except Cameron groups
have order $\leq \exp(\widetilde{O}(n^{1/3}))$

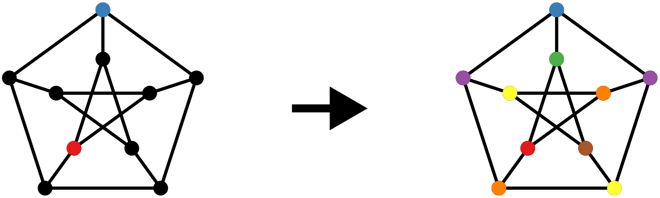
(CSFG-free proof of Cameron's theorem down to $\exp(\widetilde{O}(n^{1/3}))$)

Set S is **base** if $\mathrm{Aut}(\mathfrak{X})_{(S)} = 1$.
$\implies |\mathrm{Aut}(\mathfrak{X})| \leq \exp(\widetilde{O}(|S|))$

Want $|S| \leq \widetilde{O}(n^{1/3})$

Combinatorial relaxation:

- Fix vertices: assign unique colors (**individualize**)
- Estimate orbits: canonically **refine** coloring, examine color classes



individualize 2 vertices *propogate coloring*

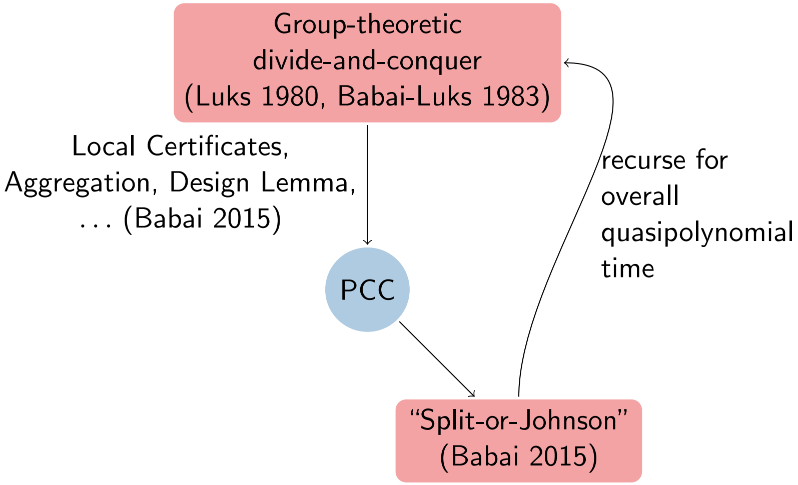
Goal: **discrete coloring** (all vertices end with unique colors)

Main result, restated

Theorem (Sun–W 2015)

Nontrivial PCCs other than $J(m, 2)$ or $H(m, 2)$ get discrete coloring from 1-WL after $\widetilde{O}(n^{1/3})$ individualizations

Babai's $\exp(\widetilde{O}(n^{1/3}))$ Algorithm for General GI



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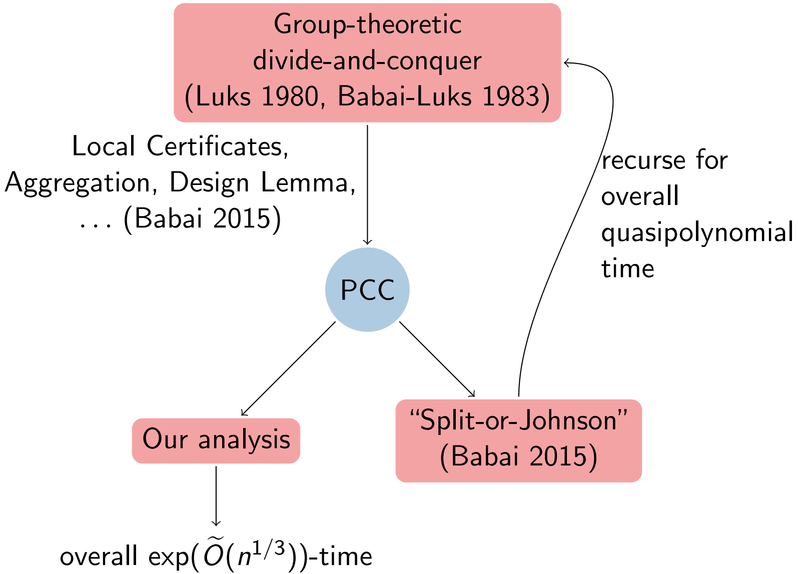
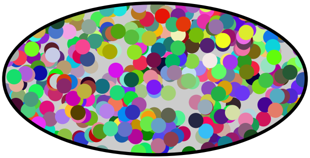


Our analysis

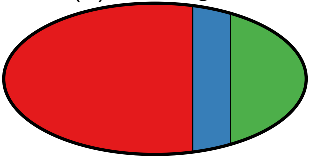
$J(m,2)$ or $H(m,2)$, or $\widetilde{O}(n^{1/3})$ indiv. gives:



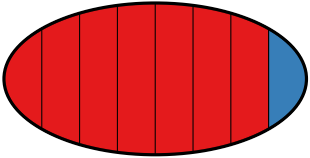
discrete coloring

Babai's "Split-or-Johnson" Lemma

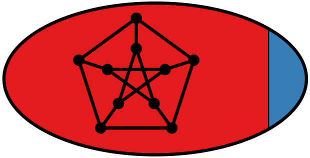
$\widetilde{O}(1)$ indiv. gives:



no dominant color, or:



equipartition of dominant color, or:



$J(m,k)$ on dominant color

Structure of primitive coherent configurations

Theorem (Sun-W)

Let $\mathfrak{X}$ be a PCC. One of these holds:

1. *every edge-color has valency $\leq n - n^{2/3}$ (no dominant color)*
2. *for some color i, we have $\lambda_i = o(\sqrt{n})$*
3. *complement of dominant color has "clique geometry"*

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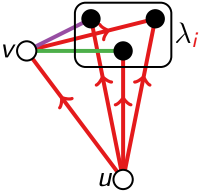

λ_i bound entails **vertex expansion**

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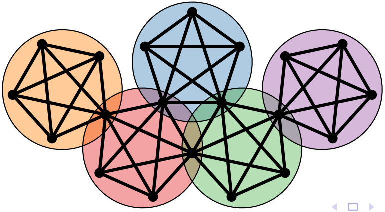
Definition

A **clique geometry** on graph G is set $\mathcal{C}$ of maximal cliques in G s.t. every edge lies in unique clique



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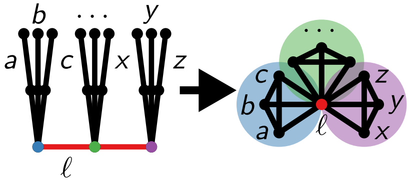

Structure of primitive coherent configurations

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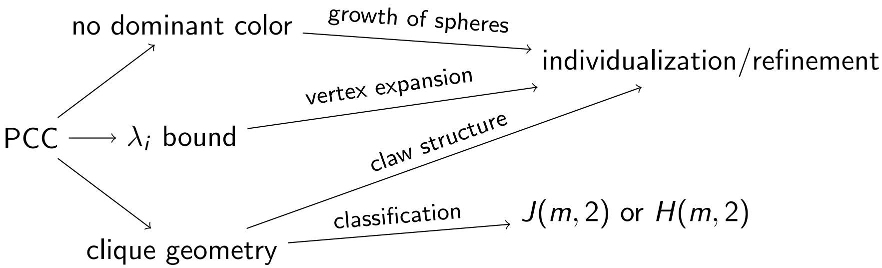
A **clique geometry** on graph G is set $\mathcal{C}$ of maximal cliques in G s.t. every edge lies in unique clique

Note: $J(m, 2)$ and $H(m, 2)$ have clique geometries

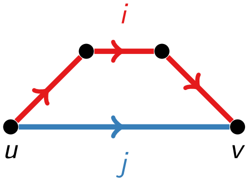
Proof sketch



When no color is overwhelmingly dominant

ρ = co-valency of highest-valency color
Assume $\rho \geq n^{2/3}$

$d = \max_{i,j} \mathrm{dist}_i(j)$ (= distance in i between color-j pair)



$\mathrm{dist}_i(j) = 3$

Lemma (Babai 1981)

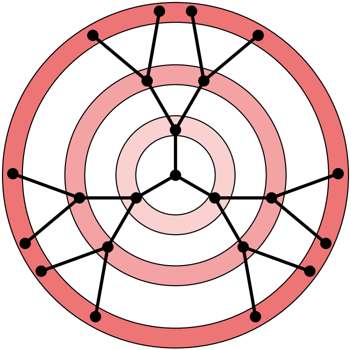
$O(nd \log n/\rho)$ individualizations suffice for discrete coloring

When no color is overwhelmingly dominant

Lemma (Babai 1981)

$O(nd \log n/\rho)$ *individualizations suffice for discrete coloring*

We estimate growth of spheres



When no color is overwhelmingly dominant

- n_i = color i valency
- $\mathrm{dist}_i(j)$ = distance in ith constituent between color-j pair
- $S_i^{(\alpha)}$ = number of vertices in α-sphere of ith constituent

Lemma (Sun-W)

Let i, j off-diagonal colors and $\delta = \mathrm{dist}_i(j)$. Then $\forall 1 \leq \alpha \leq \delta - 2$

$$S_i^{(\alpha+1)} S_i^{(\delta-\alpha)} \geq n_i n_j$$

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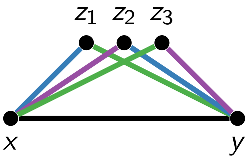
Enough when $\max n_i = \Omega(n)$

When no color is overwhelmingly dominant

What if all $n_i < \varepsilon n$? Analyze distinguishing number

$$D(u, v) = \#\{w : c(w, u) \neq c(w, v)\}$$

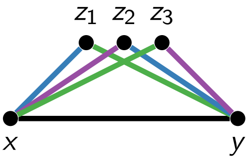


Enough: all $D(u, v)$ large

When no color is overwhelmingly dominant

What if all $n_i < \varepsilon n$? Analyze distinguishing number

$$D(u, v) = \#\{w : c(w, u) \neq c(w, v)\}$$



Enough: all $D(u, v)$ large

What if some $D(u, v) = D(c(u, v))$ small?

When no color is overwhelmingly dominant

What if some $D(u, v) = D(c(u, v))$ small?

Key tool

$D(i)$ small $\implies$ "properties vary smoothly" along i-paths

In particular, valency and distinguishing number "vary smoothly"

Proof sketch

$$
\begin{tikzcd}
 & \text{no dominant color} \arrow[r, "\text{growth of spheres}"] & \text{individualization/refinement} \\
\text{PCC} \arrow[ur] \arrow[r] \arrow[dr] & \lambda_i \text{ bound} \arrow[ur, "\text{vertex expansion}"] & \\
 & \text{clique geometry} \arrow[uur, "\text{claw structure}"] \arrow[r, "\text{classification}"] & J(m,2) \text{ or } H(m,2)
\end{tikzcd}
$$

Clique geometries in PCCs

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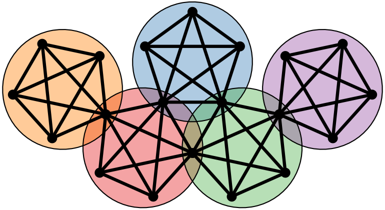
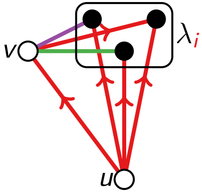
A **clique geometry** on graph G is set $\mathcal{C}$ of maximal cliques in G s.t. every edge lies in unique clique



Clique geometries in PCCs

Notation:

- $\mathfrak{X}$ = PCC
- ρ = co-valency of highest-valency constituent
- G = complement graph of highest-valency constituent
- $\lambda_i = |\mathfrak{X}_i(u) \cap G(v)|$ where $c(u, v) = i$



Clique geometries in PCCs

Theorem (Sun-W)

$(\forall c > 0)(\exists \varepsilon > 0)$ s.t. if $\lambda_i > cn^{1/2}$ and $\rho < \varepsilon n^{2/3}$,
then G has unique clique geometry.
Further, "asymptotically uniform":
if $c(u,v) = i$ in clique C then $|G(u) \cap C| = \lambda_i + O(\rho\mu/\lambda_i)$

Step 1: Find "local clique partitions" in each color

Definition

$\mathfrak{X}$ has I-local clique partitions for set I of colors if $\forall u \in \Omega\ \exists \mathcal{P}$ partition of $\mathfrak{X}_I(u)$ into maximal cliques of induced G-subgraph on $\mathfrak{X}_I(u)$

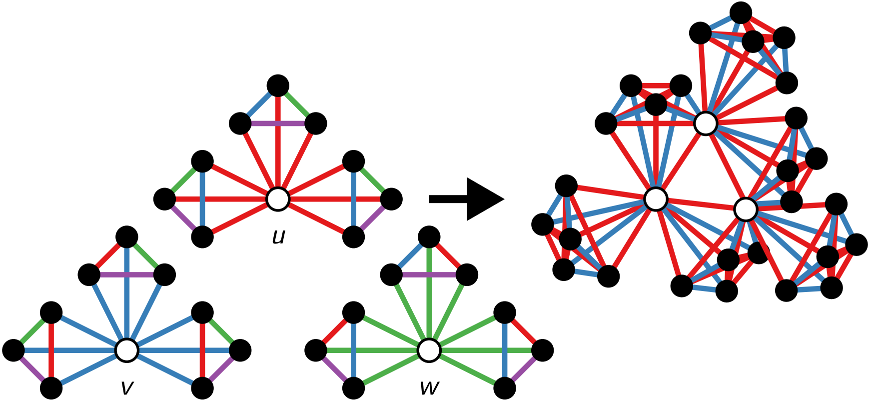
Using assumptions on ρ and λ_i, $\mathfrak{X}$ has $\{i\}$-local clique partitions $\forall i$ non-dominant by (Metsch 1991)

Finding clique geometries

Step 2: Gradually strengthen clique partitions.

2a. Ensure cliques in $\mathfrak{X}_I(u)$ maximal in G (not just induced subgraph)

2b. Ensure cliques agree at different vertices

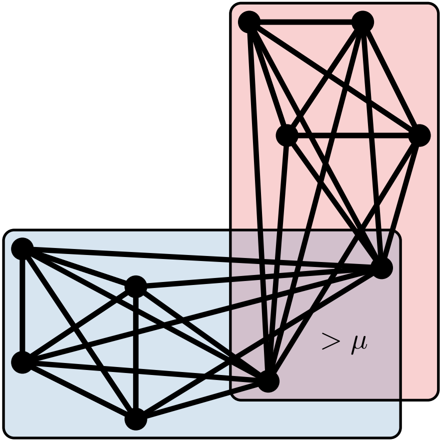


Finding clique geometries

Let $\mu = |G(u) \cap G(v)|$ for $u \not\sim v$ in G

Observation

If cliques C_1, C_2 in G have $|C_1 \cap C_2| > \mu$ then $C_1 = C_2$



Finding clique geometries

Step 3: Put it together

Theorem (Sun-W)

$(\forall c > 0)(\exists \varepsilon > 0)$ *s.t. if* $\lambda_i > cn^{1/2}$ *and* $\rho < \varepsilon n^{2/3}$,
then G *has unique clique geometry.*
Further, "asymptotically uniform":
if $c(u,v) = i$ *in clique* C *then* $|G(u) \cap C| = \lambda_i + O(\rho\mu/\lambda_i)$

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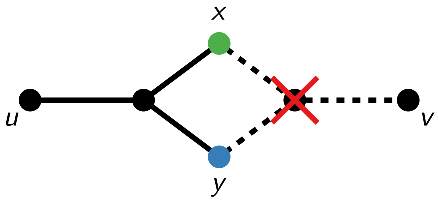
$$

Case 1: ≥ 3 cliques at each vertex

Ubiquitous 3-claws in G give gadgets:



Individualizing x, y and refining distinguishes u and v

Case 2: ≤ 2 cliques at each vertex

Theorem (Sun-W)

Suppose $\rho < \varepsilon n^{2/3}$ and $\exists$ asymptotically uniform clique geometry with ≤ 2 cliques at a vertex. Then X is one of these:

1. *trivial*
2. *Johnson $J(m, 2)$*
3. *Hamming $H(m, 2)$*
4. *rank 4 with a nonsymmetric color; symmetrization is $H(m, 2)$; $\leq n^{O(\log n)}$ automorphisms*

Summary: Primitive Coherent Configurations

Main result: PCCs except trivial, Johnson, and Hamming have $\leq \exp(\widetilde{O}(n^{1/3}))$ automorphisms

Previous best: $\exp(\widetilde{O}(n^{1/2}))$.

Proof sketch:

$$
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$$

Open Problems

Conjecture (Babai)

Nontrivial PCCs other than Cameron schemes have $\leq \exp(n^{o(1)})$ *automorphisms*

Easier cases?

- all valencies $\leq \varepsilon n$
- exists non-symmetric color

(Definitely annoying cases)

Construct infinite family of PCCs with non-uniform clique geometries?